PhD Mathematics Preparation for Economists

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Collected lecture notes and exercises

The original PDFs are combined in teaching order. Their individual page numbers restart; use the contents below or PDF bookmarks to navigate.

Contents

Week 1, Monday: KKT and envelope theorem · p. 2

Week 1, Monday: exercises · p. 9

Week 1, Tuesday: comparative statics · p. 12

Week 1, Tuesday: exercises · p. 18

Week 1, Wednesday: metric spaces and continuity · p. 20

Week 2, Monday: dynamic programming · p. 24

Week 2, Tuesday: insurance exercise · p. 27

Week 1, Monday

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Summer 2025

Topics

1. KKT Conditions and the Envelope Theorem
2. Examples
3. Exercises

1 KKT Conditions

Consider the problem

$$V(b) = \max_{\{x \in \mathbb{R}^n : g(x) \leq \mathbf{b}\}} f(x).$$

Here $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $b \in \mathbb{R}^m$. This is equivalent to writing:

$$\begin{cases} V(b) & = \max_{(x_1,\ldots,x_n)\in\mathbb{R}^n} f(x_1, \ldots, x_n) \\ st. & g_1(x_1, \ldots, x_n) \leq b_1 \\ & \vdots \\ & g_m(x_1, \ldots, x_m) \leq b_m. \end{cases}$$

The Lagrangian associated with this problem is

$$\begin{aligned} \mathcal{L}(x, \lambda) &= f(x) + \lambda \cdot (b - g(x)) \\ &= f(x_1, \ldots, x_n) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x_1, \ldots x_n)). \end{aligned}$$

Here, recall that if $u, v \in \mathbb{R}^k$ then the dot product between u and v is $u \cdot v = \sum_{i=1}^{k} u_i v_i$.

The Kuhn Tucker conditions are:

$$\begin{aligned} D_x \mathcal{L}(x, \lambda) &= \mathbf{0} \\ g(x) &\leq \mathbf{b} \\ \lambda \cdot (b - g(x)) &= \mathbf{0} \\ \lambda &\geq \mathbf{0}. \end{aligned}$$

You may translate these into its non-vector form for practice. The main theorem is the following.

Theorem 1.1.

- (Necessity) If f is continuously differentiable and if *Constraint qualification* holds then at any optimum $x^*(b)$ there exists some $\lambda \in \mathbb{R}^m$ such that the KKT conditions are satisfied.
- (Sufficiency) If f is furthermore concave in $(x_1, \ldots, x_n)$ and each g_i is convex in $(x_1, \ldots, x_n)$ and the KKT conditions are satisfied for some (x, λ) then $x^*(b) = x$. That is, x is an optimum.

We say that constraint $j \in \{1, \ldots, m\}$ is *binding* if $g_j(x^*) = b_j$. That is, at the optimum, the constraint holds with equality. We say that constraint j is *slack* otherwise. Note that if constraint j is slack then, since each $\lambda_i \geq 0$, its corresponding Lagrange multiplier λ_j must be 0 in order to satisfy the complementary slackness condition $\lambda_j(b_j - g_j(x)) = 0$.

A *nonnegativity constraint* is any constraint of the form $g_j(x) \geq 0$. The multiplies associated with nonnegativity constraints are often denoted by μ_j instead of λ_j.

1.1 The Envelope Theorem

The Envelope theorem is the following

Theorem 1.2. Suppose (x^*, λ^*) solves the KKT conditions. Then, under certain conditions, $V'(b) = \partial_\lambda \mathcal{L}(x^*, \lambda^*) = \lambda^*$.

2 Examples

Example 1: Consider

$$\begin{aligned} V(b) = &\max_{x \in \mathbb{R}} \sqrt{x} \\ &st.\ x \leq b \\ &x \geq 0, \end{aligned}$$

where $b > 0$.

This is in the general form described above with $n = 1$, $m = 2$, $f(x) = \sqrt{x}$, $g_1(x) = x$, $g_2(x) = -x$, $b_1 = b$, $b_2 = 0$. Here there is only one parameter: b. It is easy to see (since f is strictly increasing) that the maximizer is $x^*(b) = b$. So the value function is $V(b) = \sqrt{b}$. Its derivative is $V'(b) = \frac{1}{2\sqrt{b}}$. Let us now obtain the same result using the KKT conditions.

The Lagrangian is

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda(b - x) + \mu x.$$

The KKT conditions are:

$$\begin{aligned} \frac{1}{2\sqrt{x}} - \lambda + \mu &= 0 \\ x &\leq b \\ x &\geq 0 \\ \lambda_1(b - x) &= 0 \\ \lambda_2 x &= 0 \\ \mu_1 &\geq 0 \\ \mu_2 &\geq 0. \end{aligned}$$

Since f is concave and the constraints are convex, these conditions are sufficient for characterizing x^*. To solve this system, we see that $x^*(b) \neq 0$ since, for example, $x = b/2$ yields higher utility. The condition $\mu_2 x = 0$ then becomes $\mu_2 = 0$. This reduces the system to

$$\begin{aligned} \frac{1}{2\sqrt{x}} - \lambda &= 0 \\ x &\leq b \\ \mu_1(b - x) &= 0 \\ \mu_1 &\geq 0. \end{aligned}$$

Next, if $x < b$ then $\mu_1 = 0$ This would require

$$\frac{1}{2\sqrt{x}} = 0,$$

which is not possible. So it must be that $\lambda > 0$. This tells us that $x = b$. To check that $\lambda > 0$, we use the first equation to find that $\lambda = \frac{1}{2\sqrt{b}} > 0$ due to $b > 0$.

We could solve this faster by figuring out which constraints are binding and which are slack as follows. Since $\sqrt{x}$ is strictly increasing, $x^* = 0$ cannot be the optimum (in fact, we automatically know from this that $x^* = b$ but lets pretend we don't). So the constraint $x \geq 0$ is slack. Thus $\mu_2 = 0$. If $x < b$ (the first constraint is slack) then it must be that $\lambda = 0$ from which we would find that $\frac{1}{2\sqrt{x}} = 0$, which is not possible. So it must be that $x = 0$, from which we find that $\lambda^* = \frac{1}{2\sqrt{b}} \geq 0$.

The envelope theorem in this context tells us that $V'(b) = \lambda^* = \frac{1}{2\sqrt{b}}$

Example 2: Consider

$$\begin{cases} & \max_{(x_1,x_2)\in\mathbb{R}^2} x_1 x_2 \\ st. & 2x_1 + 2x_2 \leq L \\ & x_1 \geq 0 \\ & x_2 \geq 0, \end{cases}$$

Where $L > 0$.

The Lagrangian is

$$\mathcal{L}(x_1, \lambda, \mu_1, \mu_2) = x_1 x_2 + \lambda(L - 2x_1 - 2x_2) + \mu_1 x_1 + \mu_2 x_2.$$

The KKT conditions are

$$\begin{aligned} x_2 - 2\lambda + \mu_1 &= 0 \\ x_1 - 2\lambda + \mu_2 &= 0 \\ 2x_1 + 2x_2 &\leq L \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ \lambda(L - 2x_1 - 2x_2) &= 0 \\ \mu_1 x_1 &= 0 \\ \mu_2 x_2 &= 0 \\ \mu_1, \mu_2, \lambda &\geq 0. \end{aligned}$$

Again, these characterize the optimum because the objective is concave and the constraints are convex.

If either $x_1 = 0$ or $x_2 = 0$, $f(x_1, x_2) = 0$ and this can be improved by setting $x_1, x > 0$. So neither of the non-negativity constraints are binding. Thus $\mu_1, \mu_2 = 0$. If $2x_1 + 2x_2 < L$ then $\lambda = 0$ and we would now find that $x_1, x_2 = 0$, which we argued is not the optimum. So it must be that $2x_1 + 2x_2 = L$. The first two constraints now imply that $x_1 = x_2 := x$. Substituting back we find that $4x = L$, from which $(x_1, x_2) = (L/4, L/4)$. Solving for λ we find that $\lambda = L/8 > 0$.

Example 3: Consider

$$\begin{aligned} V(X) = \max_{x_1, x_2} & f_1(x_1) + f_2(x_2) \\ st.\ & x_1 + x_2 \leq X \\ & x_1, x_2 \geq 0. \end{aligned}$$

Here $X > 0$. This is a general form of a constraint optimization problem that a central planner would solve in order to coordinate production. There are two firms and f_1 and f_2 are the production functions of firms 1 and 2, respectively. These firms use the same resource type to produce a good. There is a total amount of X of the resource available. The central planner wishes to allocate this total resource amongst the two firms in order to produce the most amount of the good possible.

You should write the Lagrangian yourself for practice. The KKT conditions in this case are

$$\begin{aligned} f_1'(x_1) - \lambda + \mu_1 &= 0 \\ f_2'(x_2) - \lambda + \mu_2 &= 0 \\ x_1 + x_2 &\leq X \\ x_1, x_2 &\geq 0 \\ \lambda(X - x_1 - x_2) &= 0 \\ \mu_1 x_1 &= 0 \\ \mu_2 x_2 &= 0. \end{aligned}$$

Note that the first two conditions tell us that at an optimum where both firms are producing (so when $\mu_1 = 0 = \mu_2$), it must be that $f'(x_1) = f'(x_2)$. That is, marginal product must be equalized amongst the two firms. In this situation, $V'(X) = \lambda^* = f'(x_1^*) = f_2'(x_2^*)$.

In the preceding, the central planner knew f_1 and f_2 and coordinated production in an optimal way. The result was efficient. We can achieve the same result by introducing prices. Suppose the resource X costs a price p. Then each firm solves

$$\begin{aligned} \max_x & f_i(x) - px \\ & x \leq X. \end{aligned}$$

Since the objective is concave, interior optima satisfy $f_1'(x) = p$ and $f_2'(x) = p$. So again $f_1'(x) = f_2'(x)$. To achieve the same solution as in the planner problem, we can use $p = \lambda^*$ (then the FOCs for interior optima will coincide).

3 Exercises

Exercise 1: Consider the problem

$$\max_{x \in [a,b]} f(x).$$

Suppose f is continuously differentiable in x and let $x^* \in [a, b]$ be the optimizer (note that such an optimum exists due to the continuity of f). Show that the following *necessary conditions* hold for this problem.

- If $x^* \in (a, b)$ then $f'(x^*) = 0$.
- If $x^* = a$ then $f'(a) \leq 0$.
- If $x^* = b$ then $f'(b) \geq 0$.

If f is concave then show that the following *sufficient conditions* also hold:

- If x solves $f'(x) = 0$ and $x \in [a, b]$, then $x = x^*$.
- If $f'(a) \leq 0$ then $a = x^*$.
- If $f'(b) \geq 0$ then $b = x^*$.

Week 1, Monday

Faheem Gilani

Summer 2025

Exercise 1. An expected utility maximizer has a strictly increasing and strict concave expected utility function, $u(x)$. She has current wealth w and can buy a share, $0 < \alpha < 1$, of a risky investment X. The investment has the possibility of losing money, $Prob(X < 0) > 0$, but it has a strictly positive mean, $E\,X > 0$. The decision maker's problem is

$$\max_\alpha \; E\,u(w + \alpha X) \text{ s.t. } 0 \leq \alpha \leq 1.$$

1. Show that $f(\alpha) = E\,u(w + \alpha X)$ is strictly concave.

2. Show that $\alpha^* > 0$.

Exercise 2. An expected utility maximizer has a strictly increasing and strict concave expected utility function, $u(x)$. She has current wealth w and faces a loss $L \geq 0$ with expectation $\gamma = E\,L > 0$. For a premium $p\gamma$, $0 \leq p \leq 1$, she can insure p of her loss. This means that if $L = x$, then her wealth is $(w - p\gamma) - x + px$. Let $f(p) = E\,u((w - p\gamma) - X + pX)$. Show that the solution to the problem $\max f(p)$ s.t. $0 \leq p \leq 1$ is $p^* = 1$.

Exercise 3. There is an amount X available of one productive resource. There are n different sites where this productive resource can be used. For each i, $f_i(x_i) \geq 0$ is the output from site i if the amount of the productive resource used at site i is x_i, and it satisfies $f_i(0) = 0$, $f_i'(0) > 0$, and $f_i''(x_i) < 0$ for $x_i \geq 0$.

For $\boldsymbol{x} \geq 0$ an n-vector, we define $f(\boldsymbol{x}) = \sum_{i=1}^n f_i(x_i)$, and letting $\mathbf{1}$ be the n-vector $\mathbf{1} = (1, \ldots, 1)$, we define $g(\boldsymbol{x}) = \mathbf{1} \cdot \boldsymbol{x}$ so the problem under study is, for $X > 0$,

$$V(X) = \max_{\mathbf{x}} f(\boldsymbol{x}) \text{ subject to } g(\boldsymbol{x}) \leq X, \; \boldsymbol{x} \geq 0.$$

A. Give the non-vector form of the Kuhn-Tucker conditions for the problem $V(X)$.

B. Suppose that the solution $\boldsymbol{x}^* = \boldsymbol{x}^*(X)$ satisfies $\boldsymbol{x}^* \gg \mathbf{0}$.

   1. Show that the Kuhn-Tucker conditions imply that for all i, j, $f_i'(x_i^*) = f_j'(x_j^*)$.
   2. Give $V'(X)$.

C. Suppose that the solution $\boldsymbol{x}^* = \boldsymbol{x}^*(X)$ satisfies $x_i^* > 0$ and $x_j^* = 0$.

   1. Show that the Kuhn-Tucker conditions imply that for $f_i'(x_i^*) \geq f_j'(x_j^*)$.
   2. Give $V'(X)$.

Exercise 4. The Cobb-Douglas (CD) utility function for two goods is $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ for $0 < \alpha < 1$. A consumer with a CD utility function solves

$$V(p_1, p_2, w) = \max_{x_1, x_2} u(x_1, x_2) \text{ subject to } p_1 x_1 + p_2 x_2 \leq w,\ x_1 \geq 0,\ x_2 \geq 0,$$

where $p_1, p_2 > 0$ and $w \geq 0$.

1. Write out the Lagrangean function for this problem.
2. Write out the Kuhn-Tucker conditions for this problem.
3. Show that the solution never involves either $x_1 = 0$ or $x_2 = 0$.
4. Use the previous, simplify and show that $\boldsymbol{x}^*(\boldsymbol{p}, w)$ given by $\frac{p_1 x_1^*}{w} = \alpha$ and $\frac{p_2 x_2^*}{w} = 1 - \alpha$ solve the Kuhn-Tucker conditions.
5. The "bang for the buck" spent on good i is $\left(\partial u(\boldsymbol{x})/\partial x_i\right)/p_i$. Show that at the optimum, the bang for the buck is equalized across the two goods.
6. From the equality of the bang for the buck spent on the two goods derive the fact that the consumer's MRS (marginal rate of substitution) between the two goods at the optimum is given by the ratio of the prices for the two goods.
7. Give $V(p_1, p_2, w)$ and check that $\partial V(p_1, p_2, w)/\partial w = \lambda^*$.

Exercise 5. Consider the problem

$$\max_{x_1, x_2} u(x_1, x_2) \text{ subject to } p_1 x_1 + p_2 x_2 \leq w,\ x_1 \geq 0,\ x_2 \geq 0.$$

where u is twice continuously differentiable, strictly increasing, and concave, $p_1, p_2 > 0$ and $w \geq 0$.

1. Show that if $x^* \in \mathbb{R}^2$ solves the preceding optimization problem then $p \cdot x^* = w$.
2. Decompose the KKT conditions into cases (similar to the $1 - d$ case).

Exercise 6. The linear utility function for two goods, also known as perfect substitutes, is $u(x_1, x_2) = b_1 x_1 + b_2 x_2$ for $b_1, b_2 > 0$. A consumer with a linear utility function solves

$$V(p_1, p_2, w) = \max u(x_1, x_2) \text{ subject to } p_1 x_1 + p_2 x_2 \leq w,\ x_1 \geq 0,\ x_2 \geq 0.$$

1. Write out the Lagrangean function and the Kuhn-Tucker conditions for this problem.
2. Show that the bang for the buck determines demand behavior: what is the optimum if $\frac{b_1}{p_1} > \frac{b_2}{p_2}$, what if the inequality is reversed, and what if the inequality is an equality?
3. Verify in both cases, $\frac{b_1}{p_1} > \frac{b_2}{p_2}$ and $\frac{b_1}{p_1} < \frac{b_2}{p_2}$, that $\partial V(p_1, p_2, w)/\partial w = \lambda^*$.

Week 1, Tuesday

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Spring 2025

Topics

1. Value Function, Choice Variable, Parameters
2. 1-d comparative statics.

1 Value Functions and Maximizers

Economists usually assume that decision makers choose by maximizing some objective function. The variable that the decision maker can manipulate is called the **choice variable**. In most cases, their objective functions also depend on **parameters** that are outside of their control. In the simplest setting, such problems will often look like

$$\max_{x \in D} f(x, p),$$

f is some concave function in x, known as the **objective** function and D is some set, known as the **constraint set**.

For each p, let $x^*(p)$ be the value of the choice variable that solves the preceding problem for that p. This is often written as

$$x^*(p) = \arg\max_{x \in D} f(x, p).$$

Then the **value function** associated with the preceding problem is

$$V(p) = f(x^*(p), p).$$

We often write this as

$$V(p) = \max_{x \in D} f(x, p)$$

as well.

The value function is an "actual" function. Each parameter value p maps to a number (the value, $V(p)$) as following:

1. For a fixed p, obtain $x^*(p)$ by solving $\arg\max_{x \in D} f(x, p)$.
2. Substitute $x^*(p)$ into f to obtain $V(p) = f(x^*(p), p)$, which is just a function of p (there is no dependence on x on the right!)

2 Comparative Statics

Consider the problem

$$V(p) = \max_{x \in S} f(x, p)$$

where S is either $[a, \infty)$ or $[a, b]$ and $p \in \Theta \subset \mathbb{R}$. Comparative statics is concerned with the following two questions:

1. How does the optimal choice $x^*(p)$ depend on the parameter p?
2. How does $V(p)$ depend on p?

2.1 The dependence of $x^*(p)$ on p.

There are two approaches to this problem.

Implicit Differentiation Approach: The first approach uses implicit differentiation in order to obtain information about $\frac{dx^*(p)}{dp}$. We will illustrate this approach, being especially careful to justify the steps we are taking.

1. First, we need to either assume or show that x^* really exists. In the case when the constraint set is $[a, b]$, continuity of f guarantees that $x^*(p)$ exists for all p. However, note that even if $f(x, p)$ is concave in x then it need not have a maximizer over $[a, \infty)$. Think about the function $\log(x)$, for example.
2. In this approach, we plan on obtaining information about $\frac{dx^*(p)}{dp}$. In order to justify this, $x^*(p)$ should be a differntiable function of p (otherwise what do we mean by taking a derivative)? To do this, show or assume that $x^*(p)$ is a singleton (that is, that there is just one optimum for each p). One assumption that always guarantees this is the assumption of strict concavity: if $f(x, p)$ is strictly concave in x and if $x^*(p)$ exists for each p, then it must be a singleton.
3. The next step is to find a mathematical equation that $x^*(p)$ must satisfy. This can be done by taking a look at the necessary conditions: if $x^*(p)$ is interior for all p (that is, if $x^*(p) \in (a, b)$ in the first setup or if $x^*(p) > 0$ in the second setup), then $x^*(p)$ must satisfy $\partial_x f(x^*(p), p) = 0$.
4. Once we have the equation $\partial_x f(x^*(p), p) = 0$, we assume (and verify later) that $\frac{dx^*(p)}{dp}$ exists and implicitly differentiate both sides of this equation. This is easier to do when

we know the particular form of f, but we can also do it to the general f we have here using the chain rule. We will obtain:

$$\partial_x^2 f(x^*(p), p)\frac{dx^*(p)}{dp} + \partial_{xp} f(x^*(p), p) = 0.$$

From this, we would solve for $\frac{dx^*(p)}{dp}$ and find that

$$\frac{dx^*(p)}{dp} = -\frac{\partial_{xp} f(x^*(p), p)}{\partial_x^2 f(x^*(p), p)}.$$

This expression is well defined so long as the denominator is not zero and so long as f is twice continuously differentiable (notice the second derivatives). The former could be guaranteed by assuming that f is strictly concave in x (the existence problem is still there for unbounded sets) and the latter can be handled by assumption.

5. Notice how the expression for $\frac{dx^*(p)}{dp}$ is controlled by $\partial_{xp} f(x^*(p), p)$ and by $\partial_x^2 f(x^*(p), p)$. The former term appears in monotone comparative statics (see next method) and the latter is determined due to concavity.

Example: Consider the problem

$$V(a) = \max_{a \geq 0} \left(v(s, a) - c(s, a) \right).$$

We want to determine the sign of $\frac{dx^*(a)}{da}$.

Let $f(s, a) = v(s, a) - c(s, a)$. To use the implicit differentiation approach, we should make some assumptions. Suppose v is strictly concave in s and c is strictly convex in x. Suppose further that $\partial_{sa} v(s, a) > 0$ and $\partial_{sa} c(s, a) < 0$. We will see why we need these assumptions shortly.

Notice that the assumptions imply that f is continuously differentiable in a. Suppose $a^*(s)$ exists for all s and that $a^*(s) > 0$. Then it must be the case that $\partial_a f(s^*(a), a) = 0$. Taking the implicit derivative with respect to a yields (see the expression in step 4 also):

$$\frac{da^*(s)}{ds} = -\frac{\partial_{sa} f(s, a)}{\partial_a^2 f(s, a)}.$$

This is equivalent to

$$\frac{da^*(s)}{ds} = -\frac{\partial_{sa} v(s, a) - \partial_{sa} c(s, a)}{\partial_a^2 v(s, a) - \partial_a^2 c(s, a)}.$$

Due to our assumptions, the numerator of the preceding expression is positive (this is implied by the assumptions on the cross partials) and the denominator is negative (this is implied by the fact that v is strictly concave in a and c is strictly convex in a so that f is strictly concave in a). Therefore the entire expression (once we account for the negative sign) is strictly positive. Thus $\frac{da^*(s)}{ds} > 0$.

Please note that you don't have to memorize the general formula in step 4 of the preceding, you can get it from the first order conditions every time.

As you read this problem, ask yourself where the concavity assumption was used and why the cross partials mattered.

Monotone Comparative Statics Approach. We first define some terms and state an important theorem.

Theorem 1. Definitions:

- The function f described in the preceding is **supermodular** if for any $\theta^0, \theta \in \Theta$, $x^0, x \in X$, if $\theta^0 > \theta$ and $x^0 > x$ then
$$f(x^0, \theta^0) - f(x, \theta^0) \geq f(x^0, \theta) - f(x, \theta).$$
f is **strictly supermodular** if the preceding inequality is strict.
- f is **submodular** if the inequality in the preceding is reversed and **strictly submodular** if it is reversed and strict.

Topkis Theorem: Let $\theta^0 > \theta$. Suppose $x^*(\theta^0)$ and $x^*(\theta)$ be singleton sets. If f is supermodular then $x^*(\theta^0) \geq x^*(\theta)$. If f is strictly supermodular then the $x^*(\theta^0) \geq x^*(\theta)$. If f is submodular then $x^*(\theta^0) \leq x^*(\theta)$ and if f is strictly submodular then $x^*(\theta^0) < x^*(\theta)$

In case f is twice continuously differentiable in x and θ then we can check for super/submodularity by checking the cross partials:

Lemma 1.

1. f is supermodular if and only if $f_{x,\theta} \geq 0$. f is strictly supermodular if the preceding inequality is strict.
2. f is submodular if and only if $f_{x,\theta} \leq 0$. f is strictly submodular if the preceding inequality is strict.

The preceding can be proven using the definition of super/sub-modularity and the fundamental theorem of calculus.

The preceding gives us the following two step approach to comparative statics:

1. Check if the objective function is super or submodular.
2. If f is supermodular then $x^*(p)$ is (weakly) increasing in p. If f is submodular then $x^*(p)$ is decreasing in p

Example: Consider

$$\max_{y \geq 0} (py - c(y)) .$$

Without any assumption on $c(y)$, notice that the cross partial derivative of the objective is $1 > 0$. So the objective is supermodular. By Topkis' theorem, $y^*(p)$ (if it is singleton) is increasing.

Example: Consider

$$V(a) = \max_{a \geq 0} (v(s,a) - c(s,a)) .$$

with the assumptions that v is strictly concave in s, c is strictly convex in x, $\partial_{sa} v(s,a) > 0$, and $\partial_{sa} c(s,a) < 0$. Using just the cross partial derivatives, we find that the objective is supermodular. So by Topkis's theorem, $a^*(s)$ (if it is singleton) is increasing.

Note that we did not use concavity/convexity!

3 The dependence of V on p: The Envelope theorem.

Consider the problem

$$V(\mathbf{p}) = \max_{g(\mathbf{x},p) \geq 0} f(\mathbf{x}, \mathbf{p})$$

Suppose that f and g are continuously differentiable. Recall that the Lagrangian for this problem is

$$\mathcal{L}(\mathbf{x}, \mathbf{p}) = f(\mathbf{x}, \mathbf{p}) + \lambda \cdot g(\mathbf{x}, \mathbf{p}).$$

Suppose $x^*(\mathbf{p})$ and $\lambda^*(\mathbf{p})$ solve the optimization problem. Then

Theorem 2. (The Envelope Theorem) If V and $\mathcal{L}$ are continuously differentiable then

$$\frac{\partial V(\mathbf{p})}{\partial p_j} = \left. \frac{\partial \mathcal{L}(\mathbf{x}, \mathbf{p})}{\partial p_k} \right|_{(x^*(\mathbf{p}), \lambda^*(\mathbf{p}))} \tag{1}$$

Example For the consumer's problem, the Lagrandian was $u(x) + \lambda \cdot (w - p \cdot x) + \mu \cdot x$. So the change in the value with respect to w is λ. By Dual feasibility, $\lambda \geq 0$. This tells us that the consumer cannot be worse off when their wealth increases.

Week 1, Tuesday

Faheem Gilani

Summer 2025

Exercise 1. Suppose that a person has an original amount a of a good. By sacrificing x of it, they can produce $y = \theta g(x)$ of another good where $\theta > 0$ is a parameter (in the sense discussed in the Readiness Check) that measures the productivity of the production process. The person's utility depends on both goods, and they solve the utility maximization problem $\max_{x \geq 0} u(a - x, \theta g(x))$. Suppose that: $u(c, y) = \log(c) + y$; that $g(x) = x$, and that $a > 1/\theta$.

1. Give the KKT conditions for the maximization problem $\max_{x \geq 0} u(a - x, \theta g(x))$.
2. Solve for $x^*(\theta)$.
3. Show that $x^*(\theta)$ depends positively on θ by implicitly differentiating the KKT conditions, and check your answer by differentiating your answer to the previous part of this problem.

Exercise 2. For $p > 0$, let $V(p) = \max_{y \geq 0} \pi(y, p)$ where $\pi(y, p) = py - c(y)$ and $c''(y) > 0$. [This is the simplest version of a firm's profit maximization problem, y is quantity produced, py is revenue when y is sold at prices p, and $c(y)$ is the cost of producing the amount y.]

1. Check that for all values of the parameter p, $\partial^2 \pi(p, y)/\partial y < 0$.
2. By implicitly differentiating the KKT conditions when $y^* > 0$, show that $dy^*/dp > 0$.
3. Now show that y^* is increasing in p by using Topkis' theorem.
4. Explain the economics behind dy^*/dp being greater than 0.
5. Using the Envelope theorem, show that when $y^* > 0$, $d\pi/dp = y^*(p)$.
6. If $c'(0) = \underline{c} > 0$, give the range of prices p for which $y^*(p) = 0$.

Exercise 3. Suppose that $a \mapsto B(a)$ is a benefit function for $a \geq 0$ that satisfies $B'(a) > 0$, $B''(a) < 0$. Also suppose that $a \mapsto C(a)$ is a cost function that satisfies $C(0) = 0$, $C'(a) > 0$ and $C''(a) \geq 0$ for $a > 0$. For $\theta > 0$, let $V(\theta) = \max_{a \geq 0} B(a) - \theta C(a)$.

1. Check that for all values of the parameter θ, $\partial^2 \left(B(a) - \theta C(a) \right) / \partial a < 0$.
2. Find whether $da^*/d\theta$ is positive or negative.
3. Explain the economics behind the sign of the answer in the previous question.
4. Give conditions on $B'(0)$, $C'(0)$ and θ for which $a^* = 0$.

Exercise 4. Suppose that someone is planning for current consumption, at time $t = 0$, and future consumption, at $t = 1$. Throughout, we assume that the person has an amount X at $t = 0$, and what they do not consume at $t = 0$ can be used at $t = 1$. Consider the optimal savings problem with pure time discounting,

$$\max_{s \geq 0} U(X - s, s; \beta) = u(X - s) + \beta u(s),$$

where $u(\cdot)$ is concave and increasing and $0 < \beta < 1$ is the amount by which future consumption is discounted in utility.

1. What happens to optimal savings as β increases?
2. What happens to optimal savings as X increases?
3. The result in the previous part of the problem depends on the concavity of $u(\cdot)$, that is, on the decreasing marginal utility of consumption in any period. Why?

Week 1, Wednesday

Faheem Gilani
Summer 2025

1 Metric Spaces

A *metric space* (M, d) is a set M together with a *metric* d, which is a function $d : M \times M \to \mathbb{R}$ that satisfies the following three properties:

- (Positive Definiteness) For all $x, y \in M$, $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
- (Symmetry) For all $x, y \in M$, $d(x, y) = d(y, x)$.
- (Triangle Inequality) For all $x, y, z \in M$, $d(x, z) \leq d(x, y) + d(y, z)$.

When there is no risk of confusion, we may omit reference to d.

1. Let X be any set. Show that $d : X \times X \to \mathbb{R}$ defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

   is a metric on X (known as the *discrete* metric) and (X, d) is referred to as a *discrete metric space*.

2. Let $X = \mathbb{R}^2$ and denote $x \in X$ by $x = (x_1, x_2)$, with $x_1, x_2 \in \mathbb{R}$. Define $d_\infty : X \times X \to \mathbb{R}$ via $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. Show that d_∞ is a metric on X.

3. Prove that if (M, d) is metric space and $p, q, x, y \in M$ then $|d(p, q) - d(x, y)| \leq d(p, x) + d(q, y)$.

4. $C \subset \mathbb{R}^m$ is convex if $\lambda x + (1 - \lambda)y \in C$ whenever $x, y \in C$ and $\lambda \in [0, 1]$. Prove that the open unit ball $\{x \in \mathbb{R}^n : d(x, 0) < 1\}$ is convex.

Let (X, d) be a metric space and for any $x \in X$, let $N_\epsilon(x) = \{y \in X \mid d(y, x) < \epsilon\}$.

- A *sequence* in a metric space (M, d) is a function $f : \mathbb{N} \to M$. It is often denoted by $(p_n)_{n=1}^\infty$ where $p_n := f(n)$.

- A sequence (a_n) with values in X *converges to* $a \in X$ if for all $\epsilon > 0$, there exists $N \in N$ such that for all $n \geq N$, $a_n \in N_\epsilon(a)$.
- A sequence (a_n) with values in X is *Cauchy* if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, $d(a_n, a_m) < \epsilon$.
- A metric space (X, d) is *complete* if every cauchy sequence in X converges to a point in X.
- A set $A \subset X$ is *open* in (X, d) if for all $a \in A$, there exists an $\epsilon > 0$ such that $N_\epsilon(a) \subset A$.
- A set $F \subset X$ is *closed* in (X, d) if F^c is open in X.
- A point $x \in X$ is a *limit point* of $A \subset X$ if for all $\epsilon > 0$, there exists $y \neq x$ such that $y \in N_\epsilon(x) \cap A$.

1. Show that $\frac{1}{n}$ converges to 0 in $(\mathbb{R}, |\cdot|)$ but not in $(\mathbb{R}, d)$ where d is the discrete metric.
2. Show that a convergent sequence is Cauchy. Construct an expicit example of Cauchy sequence in a metric space that does not converge.
3. If (X, d) is a metric space then show that X is open in (X, d).
4. Find an example of a metric space in which all subsets are both open and closed.
5. Let X be a set and d the discrete metric on X. Show that a sequence a_n in X converges to $a \in X$ if and only a_n is eventually constant. That is, if and only if there exists an $N \in \mathbb{N}$ such that $a_n = a_N$ for all $n \geq N$.
6. Show that a discrete metric space is complete.
7. Let (X, d) be a metric space. Show that $F \subset X$ is closed in (X, d) if and only if F contains all of its limit points.
8. Prove that the intersection of any number of closed sets is closed and the union of any number of open sets is open.

2 The Metric Space $(\mathbb{R}, |\cdot|)$

The most well known metric space is $\mathbb{R}$ with $d(x, y) = |x - y|$. We collect releveant facts about $\mathbb{R}$ here.

Let $A \subset \mathbb{R}$. Then

- **(Upper Bound):** A is *bounded above* if there exists $b \in \mathbb{R}$ such that $b \geq a$ for all $a \in A$. In this case b is an *upper bound* of A.

- **(Least Upper Bound/ Supremum):** $s \in \mathbb{R}$ is the *least upper bound* or *supremum* of A if (i) s is an upper bound of A. (ii) if b is an upper bound of A then $s \leq A$.
- **(The Axiom of Completeness):** Every nonempty set of real numbers that is bounded above has a supremum.

1. Find an example of a set A that is bounded above but for which $\sup(A)$ is not in A.
2. Assume s is an upper bound of A. Prove that $s = \text{Sup}(A)$ if and only if for every $\epsilon > 0$, there exists an $a \in A$ satisfying $s - \epsilon < a$
3. Prove that the supremum of a set (when it exists) is unique.
4. Find (without proof) the supremum and infimum of the following sets:
   (a) $\{m/n \mid m, n \in \mathbb{N}, m < n\}$.
   (b) $\{(-1)^n \mid m, n \in \mathbb{N}\}$.
   (c) $\{n/(3n+1) \mid n \in \mathbb{N}\}$.
5. A real valued sequence (a_n) is said to be *monotone* if it is either increasing ($a_n \leq a_{n+1}$ for all n) or decreasing ($a_n \geq a_{n+1}$). Prove that if a monotone (a_n) is bounded then it converges.

3 Continuous Functions Between Metric Spaces

- A function $f : (M, d_M) \to (N, d_N)$ is **continuous** if

$$(\forall x, y \in M)(\forall \epsilon > 0)(\exists \delta > 0) d_M(x, y) < \delta \implies d_N(f(x), f(y)) < \epsilon.$$

1. Show that the following functions are continuous at the given point.
   (a) $f(x) = 3x - 5$ at $x = 2$.
   (b) $f(x) = x^2$ at $x = 3$.
   (c) $f(x) = 1/x$ at $x = 1/2$.

   Prove that f is continuous if and only if it sends convergent sequences to convergent sequences, with limits being sent to limits.
2. Find an example of a Cauchy sequence $(x_n)_{n=1}^{\infty}$ and a continuous function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an open interval, such that the sequence $f(x_n)$ is not Cauchy. Why does this not contradict the preceding?
3. Prove that if $f : (M, d_M) \to (N, d_N)$ is continuous then $f^{-1}(O)$ is open in M whenever O is open in N.

4 Compactness

- A set S in a metric space (M, d) is **compact** if, given any collection of open sets $(O_i)_{i \in I}$ such that $S \subset \cup_{i \in I} O_i$, there is a finite index set $F \subset I$ such that $S \subset \cup_{j \in F} O_j$. AKA: "every open cover has a finite subcover".
- A set S in a metric space (M, d) is *sequentially compact* if every sequence in S has a subsequence that converges to a point in S.
- Theorem: For metric spaces, compactness and sequential compactness are the same thing.

Prove the following

1. a finite subset of a metric space is compact.
2. a closed subset of a compact metric space is compact.
3. a compact set is closed and bounded.
4. $(0, 1]$ is not compact.
5. Let $f : K \to \mathbb{R}$ be a real valued continuous function defined on a compact set K.
   (a) Show that the image of K under f is compact.
   (b) Show that f is bounded.
   (c) (Extreme Value Theorem) Show that f achieves its maximum and minimum values.

Week 2, Monday

Faheem Gilani
Summer 2025

1 Finite Horizon Consumption-Saving Problem with Log Utility

Recall the problem

$$\begin{aligned} V(k_0) = \max_{(c_t,k_{t+1})_{t=0}^T} & \sum_{t=0}^{T} \beta^t u(c_t) \\ & c_t + k_{t+1} \leq f(k_t) \text{ for } T+1 \geq t \geq 0 \\ & c_t, k_t \geq 0 \text{ for } T \geq t \geq 0, \end{aligned}$$

Where u is strictly increasing, concave and with $\lim_{x\downarrow 0} u'(x) = \infty$, $\lim_{x\uparrow\infty} u'(x) = 0$, $f(0) = 0$ and $k_0 > 0$ is given.

Recall that the KKT conditions in this case yield

$$\begin{aligned} k_{T+1} &= 0 \\ c_T &= f(k_{T-1}) \\ u'(c_t) &= \beta u'(c_{t+1}) f'(k_{t+1}) \text{ for } t = 0, \ldots, T-1 \\ \beta^T u'(c_T) k_{T+1} &= 0 \end{aligned}$$

Remember that we wrote the last condition just to foreshadow the transversality condition.

We show how these equations can be solved for a particular specification. Suppose $u(c_t) = \log(c_t)$ and $f(k_t) = Ak_t^\alpha$.

1. Solve the equation for $T = 2$ in this case. You should find something like:

$$\begin{aligned} k_1 &= (1 + \beta\alpha)\beta\alpha c_0 \\ k_2 &= \beta\alpha c_1 \\ c_0 &= \frac{Ak_0^\alpha}{1 + \beta\alpha + \beta^2\alpha^2} \\ c_1 &= c_0 \beta A \alpha k_1^{\alpha-1} \\ c_2 &= c_1 \beta A \alpha k_2^{\alpha-1} \end{aligned}$$

2. Now solve the equations for general T.

2 Bellman Equation

We are now considering the infinite horizon case. We know that $k_{t+1} = f(k_t) - c_t$. Bellman's principal of optimality allows us to write the Value function as

$$V(k_0) = \max_{0 \le k_1 \le f(k_0)} \left\{ u\left(f(k_0) - k_1\right) + \beta V(k_1) \right\}$$

We know that $0 < k_1 < f(k_0)$ so the optimizer satisfies

$$-u'(f(k_0) - k_1) + \beta V'(k_1) = 0$$

By the envelope theorem,

$$V'(k_0) = u'(f(k_0) - k_1) f'(k_0).$$

1. Show that substituting this into the preceding yields the Euler equation from before.

3 Solving The Bellman Equation

In order to solve the Bellman equation, we must find both the *value function* $V(k_0)$ and the *policy function* $k_1(k_0)$.

1. How would find the value function and the policy function help solve the problem?
2. Do we need both the value function and the policy function?

Now recall the general discussion about fixed point problems of the type

$$L(x) = x$$

where L is a mapping on a metric space.

1. Let $\mathcal{C}[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$. Define the *norm* of $f \in \mathcal{C}[a,b]$ as $\|f\|_\infty = \max_{x \in [a,b]} |f(x)|$. Define the metric d on $\mathcal{C}[a,b]$ via $d(f,g) = \|f - g\|_\infty$.
   (a) Show that d is a metric on $\mathcal{C}[a,b]$.
   (b) Define $L : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ via

$$(L(g))(x) = \max_{0 \le k_1 \le f(x)} \left\{ u\left(f(x) - k_1\right) + \beta g(k_1) \right\}.$$

   Show that (under the assumptions on f and u from the preceding) that L maps $\mathcal{C}[a,b]$ to $\mathcal{C}[a,b]$.

2. We say that $L : (M, d) \to (M, d)$ is a contraction mapping on the metric space (M, d) if there exists $\beta < 1$ such that $d(L(x), L(y)) < \beta$

   (a) Suppose L is a contraction mapping on (M, d). Let $x_0 \in M$. Show that the sequence $s_1 = x_0, s_1 = L(x_0), s_3 = L^2(x_0), \ldots$ is a Cauchy sequence. Thus if M is complete, the preceding sequence converges. Hint: if $x_n > x_n$ then $d(s_n, s_m) \leq d(s_m, s_{m+1}) + \ldots + d(s_{n-1}, s_n)$. Then note that $d(s_k, s_{k+1}) = d(L^k x_0, L^k s_1)$.

   (b) If (M, d) is complete then show that $s = \lim_n s_n$ satisfies $L(s) = s$. Furthermore, s is the only element of M that satisfies this relation. Therefore s is the unique fixed point of L. Hint: Show that if L is a contraction mapping then L is continuous. Then interchange limits.

3. Consider the operator L from (b). Show that L satisfies the following two properties:

   - If $g_1 \leq g_2$ (pointwise) then $L(g_1) \leq L(g_2)$.
   - If $c \in \mathbb{R}$ then $L(g + c) \leq L(g) + \beta L(c)$ for some $0 \leq \beta < 1$. Note that addition is defined pointwise.

4. Show that the preceding two facts along with the fact that L maps continuous functions to continuous functions imply that L is a contraction mapping on $\mathcal{C}[a, b]$. HINT: Recall that $d(L(g_1), L(g_2)) = \|g_1 - g_2\|_\infty$. So you can try to show that $\|L(g_1) - L(g_2)\|_\infty < \beta\|g_1 - g_2\|_\infty$. Note the fact that $g_1 = g_2 + g_1 - g_2 \leq g_2 + \|g_1 - g_2\|_\infty$. Then use the two properties to obtain $L(g_1) \leq L(g_2) + \beta\|g_1 - g_2\|_\infty$.

5. Show that $\mathcal{C}[a, b]$ is a complete metric space (under the metric specified above. Thus the operator that defines the Bellman equation is a contraction mapping. This implies that we can solve the Bellman equation by starting with an initial function V_0 and forming the sequence $V_1 = L(V_0), V_2 = L(v_1), \ldots$, in order to obtain the fixed point of L (the Value function).

6. Instead of iterating, find the fixed point of L by guessing that it is of the form $V(x) = E + F\log(x)$ (find the E and F that make it work out). HINT: find the policy function for this guess first. You should find that $k_1 = \frac{\beta F}{1+\beta F} A k^\alpha$.

7. Find $L(\mathbf{0}), L^2(\mathbf{0}), L^3(\mathbf{0})$. Compare with the preceding.

Week 2, Tuesday

Faheem Gilani

Summer 2025

1. Solve

$$\min_{\beta \geq 0} \frac{1}{2} \sum_{i=1}^{n} (y_i - x_i \beta)^2$$

2. Solve

$$\begin{aligned} &\max_{x,y} xy \\ &p_1 x + p_2 y \leq 100 \\ &x \leq \overline{x} \\ &x, y \geq 0. \end{aligned}$$

   Your answer should consider the possible values of $\overline{x}$.

3. Solve

$$\begin{aligned} &\max_{Q_1, Q_2, K} p_1(Q_1)Q_1 + p_2(Q_2)Q_2 - b(Q_1 + Q_2) - cK \\ &Q_1 \leq K \\ &Q_2 \leq K, \end{aligned}$$

   where

$$\begin{aligned} P_1(Q_1) &= 22 - 10^{-5} Q_1 \\ P_2(Q_2) &= 18 - 10^{-5} Q_2. \end{aligned}$$

4. Let $e \in 0, 1$ and $\ell(e)$ a random variable supported on $\{0, L\}$ with $\text{Prob}(\ell(e) = L) = \pi_e$. Let u be increasing and concave. Suppose that $\frac{1-\pi_0}{1-\pi_0} < \frac{\pi_0}{\pi_1}$ and that $0 < \pi_e < 1$.

   (a) Consider

$$\begin{aligned} &\max_{e, p, B_0, B_L} p - (1 - \pi(e))B_0 - \pi(e)B_L \\ &(1 - \pi(e))u(w - p + B_0) + \pi(e)u(w - p - L + B_L) - d(e) \geq \overline{u}. \end{aligned}$$

      i. Fix $e \in \{0, 1\}$ and show using the KKT conditions that the constraint in the previous problem holds with equality.

      ii. Argue that

$$\frac{1}{u'(w - p - \ell + B_\ell)} = \lambda$$

      for each $\ell \in \{0, L\}$. Here λ is the Lagrange multiplier. Explain why this implies that $B_\ell - \ell$ is constant in ℓ.