Mathematical Microeconomic Theory with Advanced Applications

Faheem Gilani | Spring 2025

Collected lecture notes and exercises

The original PDFs are combined in teaching order. Their individual page numbers restart; use the contents below or PDF bookmarks to navigate.

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Week 1

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Spring 2025

Topics and Time allocation

1. Derivatives, Vectors, and Partial Derivatives
2. Tangent Lines and Tangent Planes
3. Concave Functions
4. Value Functions and Maximizers
5. Profit Maximization

Please see Essential Mathematics for Economic Analysis for

- Tangents and Derivatives : Section 6.2
- Simple Rules for Differentiation : Section 6.6
- Concave and Convex Functions : Section 6.9
- Implicit Differentiation : Section 7.1
- Profit Maximization Example : section 8.5 - example 1
- Partial Derivatives with Two Variables : Section 11.2
- Value Functions : Section 13.7

Introduction

Economic analysis often assumes decisions are made through optimization. This document provides key tools and examples focusing on single-variable optimization.

1-D Calculus, Vectors, and Partial Derivatives

Rules of Differentiation:

- *Chain Rule:* $h(x) = f(g(x)) \implies h'(x) = f'(g(x))g'(x)$
- *Product Rule:* $h(x) = f(x)g(x) \implies h'(x) = f'(x)g(x) + f(x)g'(x)$

- **Vectors:** R^n is the set of n-dimensional points or vectors. So, $x \in R^n$ means that $x = (x_1, \ldots, x_n)$. For example, $(1, 2) \in R^2$.
- If $x, y \in R^n$ are vectors, then their sum is defined as:

$$x + y := (x_1 + y_1, \ldots, x_n + y_n).$$

- If $x, y \in R^n$, then their **dot product** $x \cdot y$ is defined as:

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

Partial Derivatives: For $f(x_1, ..., x_n)$, the gradient is $Df(x) = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right)$ the vector of partial derivatives.

Tangent Lines and Planes

Tangent Line to $f(x)$:

$$L(x) = f(x^0) + f'(x^0)(x - x^0).$$

Tangent Plane to $f(x_1, x_2)$:

$$T(x) = f(x^0) + \frac{\partial f}{\partial x_1}(x_1 - x_1^0) + \frac{\partial f}{\partial x_2}(x_2 - x_2^0).$$

Concave Functions

A twice-differentiable function $f(x)$ is:

- *Concave* if $f''(x) \leq 0$.
- *Strictly concave* if $f''(x) < 0$.

Optimization: For concave $f(x)$, $f'(x) = 0$ is sufficient to find the maximum. Strict concavity ensures a unique maximizer.

Value Functions and Maximizers

Consider the optimization problem:

$$\max_{x \in D} f(x, p),$$

where x is the choice variable, p is a parameter, and D is the constraint set. The value function $V(p)$ represents the maximum:

$$V(p) = f(x^*(p), p),$$

where $x^*(p)$ is the optimal choice.

Profit Maximization Example

A firm maximizes profit:

$$V(p) = \max_{y \geq 0} \left(py - c(y) \right),$$

where p is price and $c(y)$ is cost.
Example with Quadratic Cost: If $c(y) = \frac{1}{2}y^2$, then:

$$V(p) = \max_{y \geq 0} \left(py - \frac{1}{2}y^2 \right)$$

Notice that for each p, the objective is concave in y due to

$$\frac{\partial^2 f}{\partial y^2} = -1 < 0.$$

Therefore, the first order conditions are sufficient for optimality. That is, to solve for $y^*(p)$ (the optimal quantity at price p), we differentiate the objective in y, set the result equal to 0 and then solve for y (as a function of p).

1. The objective is:
$$f(y,p) = py - \frac{1}{2}y^2$$

2. derivative is:
$$\frac{\partial f(y)}{\partial y} = p - y$$

3. Setting the preceding to 0 and solving for y yields:
$$y^*(p) = p.$$

To determine the value function, we plug $y^*(p)$ back into the objective function:

$$\begin{aligned} V(p) &= py^*(p) - \frac{1}{2}(y^*(p))^2 \\ &= p^2 - \frac{1}{2}p^2 \\ &= \frac{1}{2}p^2. \end{aligned}$$

Farming and Omitted Variable Bias

Consider the optimization problems:

$$\begin{aligned} V_1(c) &= \max_x \left(f(x) - cx\right), \\ V_2(c) &= \max_x \left(g(x) - cx\right), \end{aligned}$$

where $f'(x) > g'(x)$ and $f'', g'' < 0$. This represents two farmers deciding fertilizer quantities x to maximize yield, where $f(x)$ and $g(x)$ represent their yield functions. The first-order conditions imply:

$$\begin{aligned} f'(x_1^*) &= c, \\ g'(x_2^*) &= c, \end{aligned}$$

with $x_1^* > x_2^*$ due to $f'(x) > g'(x)$.

In Table 1, a statistician might estimate the fertilizer effectiveness using:

$$\frac{y_1 - y_2}{x_1 - x_2},$$

which results in a positive value, suggesting more fertilizer leads to higher yields. However, there is an omitted variable bias: both farmers choose fertilizer optimally.

Fertilizer	Yield
x_1	y_1
x_2	y_2

Table 1: Hypothetical dataset from farmer 1 and 2.

Discrimination

Consider the decision to acquire skill s:

$$V(\theta) = \max_{s \geq 0} \left(\theta f(s) - c(s) \right),$$

where $f(s)$ is the reward for skill s, $c(s)$ is the cost, and θ represents discrimination. If $\theta = 1$, the reward is $f(s)$, and if $\theta < 1$, it's reduced to $\theta f(s)$. The optimal skill choice $s^*(\theta)$ decreases as θ decreases. The first-order condition is:

$$\theta f'(s^*(\theta)) = c'(s^*(\theta)),$$

and differentiating with respect to θ yields:

$$\frac{ds^*(\theta)}{d\theta} = \frac{f'(s^*(\theta))}{c''(s^*(\theta)) - \theta f''(s^*(\theta))} > 0.$$

Thus, $s^*(\theta)$ decreases in θ, implying that greater discrimination results in less skill acquisition.

Week 2-01.21.2025
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Topics and Time allocation

1. Basic Vector Operations
2. Partial Derivatives, Gradients Reviewed
3. Tangent line, tangent planes Reviewed
4. Concavity and relevant result for optimization
5. Example Problems

Please see Essential Mathematics for Economic Analysis for

- Operations on Vectors and The Inner Product and Geometric Interpretation of Vectors : Section 15.7

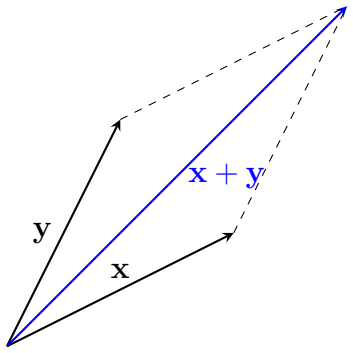
Basic Vector Operations

Let $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$ be vectors. The vector addition $\mathbf{x} + \mathbf{y}$ is defined as:

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}.$$

Geometric Representation in $\mathbb{R}^2$

The geometric interpretation of vector addition can be shown using the parallelogram rule: the sum $\mathbf{x} + \mathbf{y}$ is the diagonal of the parallelogram formed by the vectors $\mathbf{x}$ and $\mathbf{y}$.



Here, the vector $\mathbf{x}$ is added to $\mathbf{y}$ to yield $\mathbf{x} + \mathbf{y}$, shown as the diagonal of the parallelogram.

Inner Product: Example and Simple Rules

The **inner product** (or dot product) of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as:

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Example in $\mathbb{R}^3$ Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. The inner product is calculated as:

$$\mathbf{x} \cdot \mathbf{y} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32.$$

Equations for Tangent Line and Tangent Plane

Tangent Line (Single Variable)

Let $f(x)$ be a differentiable function, and let the point of tangency be $(x_0, f(x_0))$. The equation of the tangent line to $f(x)$ at $x = x_0$ is given by:

$$y = f(x_0) + f'(x_0)(x - x_0),$$

where: - $f'(x_0)$ is the slope of the tangent line at x_0.

Example If $f(x) = x^2$ and $x_0 = 1$, then:

$$f(x_0) = 1^2 = 1, \quad f'(x) = 2x, \quad f'(x_0) = 2 \cdot 1 = 2.$$

The tangent line equation is:

$$y = 1 + 2(x - 1).$$

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Tangent Plane (Two Variables)

Let $z = f(x, y)$ be a differentiable function, and let the point of tangency be $(x_0, y_0, f(x_0, y_0))$. The equation of the tangent plane to $f(x, y)$ at (x_0, y_0) is given by:

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0),$$

where: - $\frac{\partial f}{\partial x}(x_0, y_0)$ is the partial derivative of f with respect to x at (x_0, y_0), - $\frac{\partial f}{\partial y}(x_0, y_0)$ is the partial derivative of f with respect to y at (x_0, y_0).

Example If $f(x, y) = x^2 + y^2$ and $(x_0, y_0) = (1, 1)$, then:

$$f(x_0, y_0) = 1^2 + 1^2 = 2, \quad \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y.$$

At $(x_0, y_0) = (1, 1)$, we have:

$$\frac{\partial f}{\partial x}(1, 1) = 2 \cdot 1 = 2, \quad \frac{\partial f}{\partial y}(1, 1) = 2 \cdot 1 = 2.$$

The tangent plane equation is:

$$z = 2 + 2(x - 1) + 2(y - 1).$$

Recall: The Gradient

Let $f(x_1, x_2, \ldots, x_n)$ be a differentiable scalar function of n variables. The **gradient** of f, denoted by Df, is a vector that contains the partial derivatives of f with respect to each variable. It is given by:

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}.$$

Example: Let $f(x, y) = x^2 + y^2$. The gradient is:

$$Df = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}.$$

At the point $(1, 1)$, the gradient is:

$$Df(1, 1) = \begin{bmatrix} 2 \cdot 1 \\ 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

The gradient $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ points in the direction of the steepest increase in f at $(1, 1)$.

Definition: Concavity

A function $f(x)$ is said to be **concave** on an interval I if, for every $x_0 \in I$, the graph of $f(x)$ lies **below or on** the tangent line at x_0.

Mathematically, this means:

$$f(x) \leq f(x_0) + f'(x_0)(x - x_0) \quad \text{for all } x \in I.$$

Example: Let $f(x) = \ln(x)$ on $I = (0, \infty)$. For $x_0 = 1$, the tangent line is:

$$y = f(1) + f'(1)(x - 1) = 0 + 1 \cdot (x - 1) = x - 1.$$

The function $f(x) = \ln(x)$ satisfies $f(x) \leq x - 1$ for all $x > 0$, showing it is concave.

Concavity and First-Order Condition (FOC)

Result: Let $f(x)$ be a differentiable function on an interval I. If $f(x)$ is **concave** on I, then the first-order condition (FOC),

$$f'(x^*) = 0,$$

is sufficient to ensure that x^* is a global maximum of $f(x)$ on I.

Example 1

We aim to solve the problem:

$$V(L) = \max_{x_1, x_2} x_1 \cdot x_2 \quad \text{subject to} \quad 2x_1 + 2x_2 \leq L, \quad x_1 \geq 0,\, x_2 \geq 0.$$

Reformulating the Problem

From the constraint $2x_1 + 2x_2 \leq L$, we solve for x_2 in terms of x_1:

$$x_2 = \frac{L}{2} - x_1, \quad \text{for } x_1 \in \left[0, \frac{L}{2}\right].$$

Substituting $x_2 = \frac{L}{2} - x_1$ into the objective function $x_1 \cdot x_2$, the value function becomes:

$$V(L) = \max_{x_1 \in [0, L/2]} x_1 \left(\frac{L}{2} - x_1\right).$$

Unconstrained Optimization

1. **Objective Function**: The problem reduces to maximizing:

$$f(x_1) = x_1 \left(\frac{L}{2} - x_1\right) = \frac{L}{2} x_1 - x_1^2.$$

2. **First Derivative (FOC)**: Compute the derivative of $f(x_1)$ with respect to x_1 and set it equal to zero:

$$f'(x_1) = \frac{L}{2} - 2x_1.$$

Setting $f'(x_1) = 0$:

$$\frac{L}{2} - 2x_1 = 0 \quad \Longrightarrow \quad x_1^* = \frac{L}{4}.$$

3. **Second Derivative (SOC)**: Compute the second derivative to confirm concavity:

$$f''(x_1) = -2 < 0.$$

Since $f''(x_1) < 0$, the function $f(x_1)$ is concave, and $x_1^* = \frac{L}{4}$ is a maximum.

4. **Determine x_2^***: Using $x_2 = \frac{L}{2} - x_1$, substitute $x_1^* = \frac{L}{4}$:

$$x_2^* = \frac{L}{2} - \frac{L}{4} = \frac{L}{4}.$$

Solution

The optimal values are:

$$x_1^* = \frac{L}{4}, \quad x_2^* = \frac{L}{4}.$$

The maximum value of the objective function is:

$$V(L) = x_1^* \cdot x_2^* = \frac{L}{4} \cdot \frac{L}{4} = \frac{L^2}{16}.$$

Why the Constraint Holds with Equality

The constraint $2x_1 + 2x_2 = L$ must hold with equality at the optimal point for the following reasons:

- The objective function $x_1 \cdot x_2$ is maximized at the boundary of the feasible region where the constraint is binding.
- If the constraint were not binding (i.e., $x_1 + x_2 < \frac{L}{2}$), there would be unused "capacity" in the constraint, and the product $x_1 \cdot x_2$ could be increased by reallocating the resources.
- The FOC and the fact that the function $x_1 \cdot x_2$ is concave lead to the conclusion that the optimal solution occurs when the constraint is binding.

Example 2

We are tasked with maximizing the following objective function subject to the given constraint:

$$\max_{x_1, x_2} \left(2\sqrt{x_1} + \sqrt{x_2}\right), \quad \text{subject to} \quad 4x_1 + 3x_2 \leq w, \quad x_1, x_2 \geq 0.$$

Observation: The Constraint Holds with Equality

At the optimal solution, we must have the constraint $4x_1 + 3x_2 = w$ holding with equality, since otherwise, we would have slack in the constraint, which would allow us to increase x_1 or x_2 further to improve the objective function. This follows from the fact that the objective function $2\sqrt{x_1} + \sqrt{x_2}$ is strictly increasing in both x_1 and x_2. Therefore, the constraint must bind with equality, and we can express x_2 as a function of x_1:

$$x_2 = \frac{w - 4x_1}{3}.$$

Reducing the Problem to a Single Variable

Substitute $x_2 = \frac{w-4x_1}{3}$ into the objective function to reduce the problem to a single-variable optimization:

$$f(x_1) = 2\sqrt{x_1} + \sqrt{\frac{w - 4x_1}{3}}.$$

Now, we need to check whether the resulting objective function is concave and then solve for optimal

$$x^* = (x_1^*, x_2^*)$$

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Week 2, Tuesday

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Spring 2025

Topics

1. Super/Sub-modularity
2. Topkis' theorem
3. Monopolist vs Society
4. Training Workers

1 Setup

We are considering the problem

$$V(\theta) = \max_{x \in X} f(x, \theta)$$

where $X \subset \mathbb{R}$, $\theta \in \Theta \subset \mathbb{R}$. Let $x^*(\theta)$ be the maximizer of this problem. We suppose in this class (almost always) that $x^*(\theta)$ is a singleton.

In the profit maximization problem from last week, $\theta = p$, the price, $x = y$, the supply choice, $f(x, p) = py - c(y)$ is the objective function (c is the cost function) and $y^*(p)$ is the supply at price p.

2 Topkis' Theorem

We are interested in the understanding how x^* varies with respect to θ. In particuluar, if $\theta^0 > \theta$ then how does $x^*(\theta^0)$ compare with $x^*(\theta)$? Monotone comparative statics is concerned with the situation where x^* is monotonic in θ. Topkis' theorem tells us what happens from basic information about the objective function.

Theorem 1. Definitions:

- The function f described in the preceding is **supermodular** if for any $\theta^0, \theta \in \Theta$, $x^0, x \in X$, if $\theta^0 > \theta$ and $x^0 > x$ then

$$f(x^0, \theta^0) - f(x, \theta^0) \geq f(x^0, \theta) - f(x, \theta).$$

f is **strictly supermodular** if the preceding inequality is strict.

- f is **submodular** if the inequality in the preceding is reversed and **strictly submodular** if it is reversed and strict.

Topkis Theorem: Let $\theta^0 > \theta$. Suppose $x^*(\theta^0)$ and $x^*(\theta)$ be singleton sets. If f is supermodular then $x^*(\theta^0) \geq x^*(\theta)$. If f is strictly supermodular then the $x^*(\theta^0) \geq x^*(\theta)$. If f is submodular then $x^*(\theta^0) \leq x^*(\theta)$ and if f is strictly submodular then $x^*(\theta^0) < x^*(\theta)$

In case f is twice continuously differentiable in x and θ then we can check for super/submodularity by checking the cross partials:

Lemma 1.

1. f is supermodular if and only if $f_{x,\theta} \geq 0$. f is strictly supermodular if the preceding inequality is strict.
2. f is submodular if and only if $f_{x,\theta} \leq 0$. f is strictly submodular if the preceding inequality is strict.

The preceding can be proven using the definition of super/sub-modularity and the fundamental theorem of calculus.

3 Monopolist vs Society

Consider a (mostly) downward sloping demand curve $p(q)$. The consumer surplus is $CS(q) = \int_0^q p(t)dt - p(q)q$. If $q^0 > q$, $CS(q^0) > CS(q)$.

A monopolist is interested in maximizing their revenue. So they solve:

$$\max_{q \geq 0} \left(qp(q) - c(q) \right).$$

Society wishes to maximize the sum of consumer surplus and the monopolist's revenue. So Society solves:

$$\max_{q \geq 0} \left((qp(q) - c(q)) + CS(q) \right).$$

We wish to compare the optimum of these problems. To this end, consider the auxillary function:

$$f(q, \theta) = ((qp(q) - c(q)) + \theta CS(q))$$

and the associated problem:

$$\max_{q \geq 0} f(x, \theta).$$

Then note that $q^*(0)$ is the monopolist optimum and $q^*(1)$ is society's optimum. Assuming differentiability, we see that

$$\partial_{x,\theta} f(q,\theta) = CS'(q) > 0,$$

where the inequality follows from the second sentence in the example. Therefore f is supermodular and by Topkis' theorem we have that $q^*(1) > q^*(0)$.

Even without differentiability we can show that f is supermodular and conclude the same thing. Let $q^0 > q$ and $\theta^0 > \theta$. We want to show that

$$f(q^0,\theta^0) - f(q,\theta^0) \geq f(q^0,\theta) - f(q,\theta).$$

This is equivalent to showing that

$$q^0 p(q^0) - c(q^0) + \theta^0 CS(q^0) - qp(q) + c(q) - \theta^0 CS(q) \geq q^0 p(q^0) + \theta CS(q^0) - c(q^0) - qp(q) + c(q) - \theta CS(q).$$

After canceling, this is equivalent to showing that

$$\theta^0 \left(CS(q^0) - CS(q)\right) \geq \theta \left(CS(q^0) - CS(q)\right),$$

which is true due to the fact that consumer surplus is increasing in q and $\theta^0 > \theta$. Thus the objective is supermodular and we find again that $q^*(1) \geq q^*(0)$.

4 Training Workers

A firm wishes to provide their workers with a level of education x in order to solve

$$\max_{x \geq 0} \Pi(x),$$

where Π is their profit. Since a training level of x also provides the worker's with a benefit $B(x)$ to the worker and society, society wishes to solve

$$\max_{x \geq 0} \left(\Pi(x) + B(x)\right).$$

As before, we can combine these problems into one problem:

$$\max_{x \geq 0} \left(\Pi(x) + \theta B(x)\right).$$

The objective for this problem is supermodular so again we have by Topkis theorem that $x^*(1) \geq x^*(0)$. That is, the firm provides a lower level of education when only taking their profits into account.

5 Submodularity

An individual is solving

$$\max_{x \geq 0} NB(x)$$

where $NB(x)$ is the net benefit to the individual from taking action level x. Society wishes to solve

$$\max_{x \geq 0} \left(NB(x) - C(x) \right)$$

where $C(x)$ is the cost of the action level to society. This could be an example of firm choosing an amount of harmful atmospheric emissions (through their production). We can combine these into a single problem with a submodular objective find, due to Topkis' theorem, that $x^*(1) \leq x^*(0)$.

Week 3

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Spring 2025

Topics

1. Necessary and Sufficient Conditions for Single Variable Optimization
2. Comparative statics.

1 Necessary and Sufficient Conditions for Single Variable Optimization

Consider the problem

$$\max_{x \in [a,b]} f(x).$$

Suppose f is continuously differentiable in x and let $x^* \in [a, b]$ be the optimizer (note that such an optimum exists due to the continuity of f). Then the following *necessary conditions* hold.

- If $x^* \in (a, b)$ then $f'(x^*) = 0$.
- If $x^* = a$ then $f'(a) \leq 0$.
- If $x^* = b$ then $f'(b) \geq 0$.

If f is concave then the following *sufficient conditions* also hold:

- If x solves $f'(x) = 0$ and $x \in [a, b]$, then $x = x^*$.
- If $f'(a) \leq 0$ then $a = x^*$.
- If $f'(b) \geq 0$ then $b = x^*$.

Consider the problem

$$\max_{x \geq a} f(x).$$

Suppose f is continuously differentiable in x and that $x^* \in [a, \infty)$ exist. Then the following *necessary conditions* hold.

- If $x^* > 0$ then $f'(x^*) = 0$.
- If $x^* = a$ then $f'(a) \leq 0$.

If f is concave then we additionally have the following *sufficient conditions*:

- If $f'(x) = 0$ and $x \geq 0$ then $x = x^*$.
- If $f'(0) \leq 0$ then $0 = x^*$.

2 Comparative Statics

Consider the problem

$$V(p) = \max_{x \in S} f(x, p)$$

where S is either $[a, \infty)$ or $[a, b]$. Comparative statics is concerned with the following two questions:

1. How does the optimal choice $x^*(p)$ depend on the parameter p?
2. How does $V(p)$ deoend on p.

2.1 The dependence of $x^*(p)$ on p.

There are two approaches to this problem.

Implicit Differentiation Approach: The first approach uses implicit differentiation in order to obtain information about $\frac{dx^*(p)}{dp}$. We will illustrate this approach, being especially careful to justify the steps we are taking.

1. First, we need to either assume or show that x^* really exists. In the case when the constraint set is $[a, b]$, continuity of f guarantees that $x^*(p)$ exists for all p. However, note that even if $f(x, p)$ is concave in x then it need not have a maximizer over $[a, \infty)$. Think about the function $\log(x)$, for example.
2. In this approach, we plan on obtaining information about $\frac{dx^*(p)}{dp}$. In order to justify this, $x^*(p)$ should be a function of p (otherwise what do we mean by taking a derivative)? To do this, show or assume that $x^*(p)$ is a singleton (that is, that there is just one optimum for each p). One assumption that always guarantees this is the assumption of strict concavity: if $f(x, p)$ is strictly concave in x and if $x^*(p)$ exists for each p, then it must be a singleton.
3. The next step is to find a mathematical equation that $x^*(p)$ must satisfy. This can be done by taking a look at the necessary conditions: if $x^*(p)$ is interior for all p (that is, if $x^*(p) \in (a, b)$ in the first setup or if $x^*(p) > 0$ in the second setup), then $x^*(p)$ must satisfy $\partial_x f(x^*(p), p) = 0$.
4. Once we have the equation $\partial_x f(x^*(p), p) = 0$, we assume (and verify later) that $\frac{dx^*(p)}{dp}$ exists and implicitly differentiate both sides of this equation. This is easier to do when

we know the particular form of f, but we can also do it to the general f we have here using the chain rule. We will obtain:

$$\partial_x^2 f(x^*(p), p)\frac{dx^*(p)}{dp} + \partial_{xp} f(x^*(p), p) = 0.$$

From this, we would solve for $\frac{dx^*(p)}{dp}$ and find that

$$\frac{dx^*(p)}{dp} = -\frac{\partial_{xp} f(x^*(p), p)}{\partial_x^2 f(x^*(p), p)}.$$

This expression is well defined so long as the denominator is not zero and so long as f is twice continuously differentiable (notice the second derivatives). The former could be guaranteed by assuming that f is strictly concave in x (the existence problem is still there for unbounded sets) and the latter can be handled by assumption.

5. Notice how the expression for $\frac{dx^*(p)}{dp}$ is controlled by $\partial_{xp} f(x^*(p), p)$ and by $\partial_x^2 f(x^*(p), p)$. The former term appears in monotone comparative statics (see next method) and the latter is determined due to concavity.

Example: Consider the problem

$$V(a) = \max_{a \geq 0} \left(v(s, a) - c(s, a) \right).$$

We want to understand $\frac{dx^*(a)}{da}$.

Let $f(s, a) = v(s, a) - c(s, a)$. To use the implicit differentiation approach, we should make some assumptions. Suppose v is strictly concave in s and c is strictly convex in x. Suppose further that $\partial_{sa} v(s, a) > 0$ and $\partial_{sa} c(s, a) < 0$. We will see why we need these assumptions shortly.

Notice that the assumptions imply that f is continuously differentiable in a. Suppose $a^*(s)$ exists for all s and that $a^*(s) > 0$. Then it must be the case that $\partial_a f(s^*(a), a) = 0$. Taking the implicit derivative with respect to a yields (see the expression in step 4 also):

$$\frac{da^*(s)}{ds} = -\frac{\partial_{sa} f(s, a)}{\partial_a^2 f(s, a)}.$$

This is equivalent to

$$\frac{da^*(s)}{ds} = -\frac{\partial_{sa} v(s, a) - \partial_{sa} c(s, a)}{\partial_a^2 v(s, a) - \partial_a^2 c(s, a)}.$$

Due to our assumptions, the numerator of the preceding expression is positive (this is implied by the assumptions on the cross partials) and the denominator is negative (this is implied by the fact that v is strictly concave in a and c is strictly convex in a so that f is strictly concave in a). Therefore the entire expression (once we account for the negative sign) is strictly positive. Thus $\frac{da^*(s)}{ds} > 0$.

Please note that you don't have to memorize the general formula in step 4 of the preceding, you can get it from the first order conditions every time.

As you read this problem, ask yourself where the concavity assumption was used and why the cross partials mattered.

Monotone Comparative Statics Approach. We are considering again the maximization problem from the previous section. For this approach, please see the previous lecture notes for a statement of Topkis theorem. The idea here is to argue that the objective is super or submodular and directly apply Topkis.

1. Check if the objective function is super or submodular.
2. If f is supermodular then $x^*(p)$ is (weakly) increasing in p. If f is submodular then $x^*(p)$ is decreasing in p

Example: Lets consider the profit maximization problem again:

$$\max_{y \geq 0} \left(py - c(y) \right).$$

Without any assumption on $c(y)$, notice that the cross partial derivative of the objective is $1 > 0$. So the objective is supermodular. By Topkis' theorem, $y^*(p)$ (if it is singleton) is increasing.

This seems like a more reasonable economic statement. Previously, we stated that $y^*(p)$ is increasing under the assumption of convex costs. Convexity is a mathematical requirement and seems awkward in an economic context. But, thanks to monotone comparative statics, we can get the same result without any assumptions on cost!

Something Cool: Before we get too happy, lets check this against the implicit differentiation approach. This approach seems to claim that it does not matter what the cost function $c(y)$ looks like. But the implicit differentiation approach would directly tell us that $\frac{dy^*(p)}{dp} = 1/c''(y^*(p))$. This can only be positive if the denominator is positive! So can we truly just use any function we want? Well it turns out that no matter what function we use, if the objective is twice continuously differentiable and if the maximizer exists then the second derivative will be negative at the optimum (this is the second derivative test for classifying optima). This also resolves the confusion you may have had with the expression in step 4 above: you might have been thinking: "wait...if MCS is a real thing then shouldn't $\frac{dx^*(p)}{dp}$ and $\partial_{xp} f(x^*(p), p)$ have the same sign? Is MCS fake news??" The answer is: "MCS IS awesome! They DO have the same sign because at the optimum the denominator is of that expression is guaranteed to be negative for twice continuously differentiable functions! So $\frac{dx^*(p)}{dp}$ and $\partial_{xp} f(x^*(p), p)$ have the same sign! Furthermore, if we had kept the second

derivative test in mind when looking at equation 4 then we would have been able to come up with the definition of super/sub-modularity on our own!!!

Example: Consider again the problem

$$V(a) = \max_{a \geq 0} \left(v(s,a) - c(s,a) \right).$$

with the assumptions that v is strictly concave in s, c is strictly convex in x, $\partial_{sa} v(s,a) > 0$, and $\partial_{sa} c(s,a) < 0$. Using just the cross partial derivatives, we find that the objective is supermodular. So by Topkis's theorem, $a^*(s)$ (if it is singleton) is increasing.

Note that we did not use concavity/convexity!

02.05.2024, Tuesday

Faheem Gilani, Hilal Taşkan
Spring 2025

Topics

1. Envelope Theorem and Applications
2. Cournot Competition

1 Value Functions

The value function gives the maximum values of the objective function as a function of the parameter(s). Through the chain rule and the first order conditions, we can determine the derivative of the value function in a particular parameter. To this end, consider the following maximization problem with choice variable x, and parameter α.

$$V(\alpha) := \max_{x \geq 0} f(x, \alpha). \tag{1}$$

The first-order necessary condition is:

$$f_x(x, \alpha) = 0. \tag{2}$$

If these conditions are sufficient, such as in the case where f is concave, we find that the solution x to this equation (so long as it is feasible), is the optimizer. In other words,

$$x = x^*(\alpha). \tag{3}$$

If we substitute these solutions into the objective function, we obtain the value function:

$$V(\alpha) = f(x^*(\alpha), \alpha), \tag{4}$$

If we differentiate V with respect to α we obtain (through the chain rule):

$$V'(\alpha) = f_x(x^*(\alpha), \alpha)\frac{dx^*(x,\alpha)}{d\alpha}\bigg|_{(x^*(\alpha),\alpha)} + f_\alpha(x,\alpha)\bigg|_{(x^*(\alpha),\alpha)}. \tag{5}$$

However, from the first-order conditions, we know $f_x(x^*(\alpha), \alpha) = 0$. Therefore, the first term disappears, and we find that

$$V'(\alpha) = f_\alpha(x,\alpha)\big|_{(x^*(\alpha),\alpha)}. \tag{6}$$

Note that α enters the value function (Equation 4) in two places: indirectly through $x^*(\alpha)$ and directly through α. and one indirectly. Equations (5) and (6) show that, at the optimum, only the direct effect of α on the objective function matters for calculating the derivative of the value function. Here is the formal statement of this result.

Theorem 1. (The Envelope Theorem) Consider

$$V(\alpha) = \max_{x \in S} f(x,\alpha),$$

where $S = [a, \infty)$ or $S = [a, b]$. Suppose that V and f are continuously differentiable and suppose that $x^*(\alpha)$ exists and that it is in the interior of S. Then

$$V'(\alpha) = \partial_\alpha f(x,\alpha)\big|_{x=x^*(\alpha),\alpha=\alpha}.$$

2 A Two-Player Game: Arms Race

An arms race takes place when two or more nations continuously expand and enhance their military capabilities in pursuit of strategic and political dominance over each other. One of the most notable and costly arms races in history was the Cold War rivalry between the United States and the Soviet Union. However, similar competitions have occurred at various times, often leading to severe consequences.

Single Person Game: Let

$$\begin{aligned} f(x,\theta) &= B(x) - \theta C(x) \\ B(x), B'(x) > 0, &\quad B''(x) < 0, B(0) = 0 \\ C'(x) > 0, &\quad C''(x) > 0, \end{aligned}$$

and consider the problem

$$V(\theta) = \max_{x \geq 0} f(x, \theta).$$

In this problem, $B(x)$ is the benefit to the country from allocating x units of resources to arms development and $C(x)$ is the corresponding cost. Let $x^*(\theta)$ be the optimizer for a given θ. If $x^*(\theta) > 0$ then the envelope theorem tells us that

$$V'(\theta) = -C(x^*(\theta)).$$

The latter is negative due to the assumptions on c. Thus, the value is decreasing in θ.

Two-Person Version (Best Response Analysis): Imagine now that there are two countries. If country 1 and 2 dedicates resources (x_1, x_2), respectively, to the development of nuclear arms then country i obtains a payoff of

$$u_i(x_i, x_j) = B(x_i - x_j) - \theta C(x_i).$$

We say that (x_1^*, x_2^*) is a Nash equilibrium of this game (not formally defined yet) if

$$x_1^* \in \arg\max \max_{x_1 \geq 0} u_1(x_1, x_2^*)$$
$$x_2^* \in \arg\max \max_{x_2 \geq 0} u_2(x_1, x_2^*)$$

Due to concavity, we can characterize all interior Nash equilibria through the equations

$$B'(x_1^* - x_2^*) - \theta C'(x_1^*) = 0$$
$$B'(x_2^* - x_1^*) - \theta C'(x_2^*) = 0.$$

Due to the assumptions in this problem, one can show that the Nash equilibrium exists and that it involves both countries doing the same action: that is, $x_1^* = x_2^* = x^*$ for $x^* > 0$ satisfying $\theta^{-1} B'(0) = C'(x^*)$. Differentiating implicitly, we find that $\frac{dx^*}{d\theta} = -\frac{\theta^{-2} B'(0)}{C''(x^*)} \leq 0$.

In that case, the value to player i is $-\theta C(x^*)$. Differentiating with respect to θ and using $\theta^{-1} B'(0) = C'(x^*)$ and the equation for $x^{*'}(\theta)$, we find

$$-C(x^*) - \theta C'(x^*) \frac{dx^*}{d\theta} = -C(x^*) + \frac{\theta^{-2} B'(0)^2}{C''(x^*)}.$$

The latter is positive if

$$-C(x^*) + \frac{\theta^{-2} B'(0)^2}{C''(x^*)} \geq 0 \iff B'(0)^2 \geq \theta^2 C''(x^*) C(x^*)$$

If $C(x) = \frac{1}{2} x^2$ then $x^* = \frac{B'(0)}{\theta}$ So

$$\theta^2 C''(x^*) C(x^*) = 0 \leq B'(0),$$

so in this case each player's value is increasing in θ (contrast this with the single player case).

3 Matching Pennies

Two people (e.g. Sherlock Holmes and I) choose, simultaneously, whether to show the Head or the Tail of a coin. If they show the same side, person 2 pays person 1 a dollar; if they show different sides, person 1 pays person 2 a dollar. Each person cares only about the amount of money they receive and naturally prefers to receive more rather than less.

A strategic game that models this situation is represented in the following payoff matrix:

	Head	Tail
Head	(1, -1)	(-1, 1)
Tail	(-1, 1)	(1, -1)

In this representation of the players' preferences, the payoffs are equal to the amounts of money involved. We could equally well use another representation—for example, replacing each 1 with 2 and each -1 with -2. We will show later that the unique Nash equilibria of this game is for both players to randomly select Heads/Tails with equal probability.

4 Second-Mover Advantage in a Sequential Game

If the game is played sequentially—where one player moves first and the other observes and then responds—the second player gains an advantage for the following reasons:

A) The first player must choose either Head or Tail without knowing how the second player will react, making their decision uncertain.

B) Since the second player can observe the first player's choice, they can always select the best possible response. Given that player 2 benefits from choosing the opposite side of player 1, they can always ensure a win by picking the alternative outcome.

C) As a result, the second player always secures a victory in this sequential setting, turning it into a zero-sum game where the second-mover has a perfect strategy for winning.

5 Cournot Duopoly Competition

Two firms, labeled 1 and 2, produce the same product and compete for the same market of potential customers. The firms simultaneously choose their production quantities, q_1 and q_2, respectively. The market price of the product is determined by the market demand, which is identical for both firms.

Assume that when the supply is $q_1 + q_2$, the price per unit is given by:

$$P(Q) = a - b(q_1 + q_2),$$

where $a, b > 0$.

Assume that product costs are linear, with marginal cost $c_1 > 0$ for firm 1 and $c_2 > 0$ for firm 2. Then firm i's profit is

$$\pi_i(q_1, q_2) = q_i(a - b(q_1 + q_2)) - cq_i$$

6 Finding a Nash Equilibrium

A Nash equilibria for the Cournot game is a pair (q_1^*, q_2^*) that satisfies

$$q_1^* \in \arg\max_{q_1 \geq 0} \pi_1(q_1, q_2^*)$$
$$q_2^* \in \arg\max_{q_1 \geq 0} \pi_2(q_1^*, q_2)$$

Since each profit function is strictly concave in own production, all interior Nash equilibria are characterized by the first order conditions:

$$a - bq_2^* - 2bq_1^* = c$$
$$a - bq_1^* - 2bq_2^* = c$$

Solving this for q_1^* and q_2^* yields

$$q_1^* = \frac{a - c}{3b}$$
$$q_2^* = \frac{a - c}{3b}.$$

02.06.2024, Thursday

Faheem Gilani, Hilal Taşkan

Spring 2025

Topics

1. Risky Ventures
2. Risk Aversion-Jensen's inequality.
3. Risk Sharing
4. Insuring Against Losses

Helpful Readings

1. Probability and Random Variables: Chapter 2 of NS, Mathematical Statistics Section, Page 64 of the 10th edition.
2. Choice under uncertainty: Chapter 7, pages 202-208.

1 Risky Ventures

An individual has wealth w and can buy a risky venture X (a gamble) at price p. X is a random variable.

- Suppose the venture promises to double the amount you invest with probability $1/2$ and destroy half the amount you invest with probability $1/2$. Then $X = 2$ with probability $1/2$ and $X = 1/2$ with probability $1/2$. The expected value of this random variable is $2(1/2) + 1(1/2) = 1.25$. Suppose that the price of this venture is also 1.25 so that the venture is *fair*. The decisionmaker therefore faces the random variable $w + X - p$ if they choose to buy the venture. The expected value of this random variable is w (their initial wealth). We will see later (as a consequence of Jensen's inequality) that a *risk averse* (to be defined later) individual will not buy this venture. In general, a risk averse individual will not accept fair gambles.
- Suppose now that $E(X) > p$ and that the individual has the option to buy a proportion α of the venture. So they will spend αp in order to face the random variable $w+\alpha(X-p)$. The expectation of this random variable is $E(w+\alpha(X-p)) = w+\alpha(E(X)-p) > w$. We will see later that a risk averse individual will buy some portion $\alpha > 0$ of this

venture. This is the basis of risk sharing. Holland and England gained vast Naval power due to real life implementation of this risk sharing idea—they developed the stock market.

2 Risk Aversion

A decisionmaker has Bernoulli utility u and wishes to maximize their expected utility. We say that the decisionmaker is *risk averse* if for all gambles X, they prefer the sure amount $E(X)$ to the gamble X. Jensen's inequality tells us that if u is concave then the decisionmaker is risk aversion:

Theorem 2.1. If $f''(x) < 0$ and X is a random variable with $\text{Var}(X) > 0$ then

$$E(f(X)) < f(E(X)).$$

In order to prove this, we first prove an auxillary lemma:

Lemma 2.1. Let $L(x) = a + bx$. Then $E(L(X)) = L(E(X))$. Equivalently, $E(a + bX) = a + bE(X)$.

Proof. We prove this for discrete random variables: If X is a discrete random variable taking on values $x_1, x_2 \dots x_n$ with respective probabilities $p_1, p_2, \dots$ then $EX = \sum_{i=1}^{n} p_i x_i$. So $E(L(X)) = \sum_{i=1}^{n} (a + bx_i)p_i = \sum_{i=1}^{n} ap_i + b\sum_{i=1}^{n} x_i p_i = a + bE(X)$. □

Here is the proof for Jensen's inequality:

Proof. The equation of the tangent line to f at $E(X)$ is $L(x) = f(E(X)) + f'(E(X))(X - E(X))$. Since f is concave, for all $x \neq E(X)$, $L(x) > f(x)$. This implies that $E(L(X)) > Ef(X)$. From the lemma, we know that $E(L(X)) = L(E(X))$. Since the tangent line takes the same value as the function does at the point of tangency, $L(E(X)) = f(E(X))$. This establishes that $f(E(X)) > Ef(X)$. □

For example, if $X = 160$ with probability $1/2$ and $X = 60$ with probability $1/2$ then accepting this gamble would yield expected utility $1/2u(160)+1/2u(60)$. Meanwhile $E(X) = 1/2(160) + 1/2(60) = 80 + 30 = 110$ so $u(E(X)) = u(110)$. Jensen's inequality states that $1/2u(60) + 1/2u(110) \leq u(110)$. Again: an individual with concave utility prefers the sure amount $E(X)$ to the gamble X.

This leads to the following thought experiment. In the preceding example, we know that the decision maker would strictly prefer having 110 for sure than to take the gamble X. Would the decision maker also strictly prefer (say) having 109 to taking the gamble X? What

amount $c_X < 110$ would make them indifferent between having c_X for sure and taking the gamble X? This leads to the following.

The *certainty equivalent* of a gamble X is the number c_X satisfying $u(c_X) = E(u(X))$. By Jensen's inequality, $E(u(X)) < u(E(X))$ so $u(c_X) < u(E(X))$. Thus an insurance company could capture revenue up to an amount that makes $u(E(X)) - u(c_X) = 0$ from individuals.

3 Risk Sharing

Consider a decisionmaker with concave and strictly increasing Bernoulli utility u and initial wealth w. Suppose that the decisionmaker wishes to choose an amount $\alpha \in [0, 1]$ to invest in a risky venture X that costs p in order to maximize their expected utility. Thus the decisionmaker wishes to maximize

$$\max_{0 \leq \alpha \leq 1} E\left(u\left(w + \alpha(X - p)\right)\right).$$

Suppose $E(X) > p$.

Let $f(\alpha) := E\left(u\left(w + \alpha(X - p)\right)\right)$. Then

- f is concave in α
- $f'(0) > 0$.

The first is due to

$$f''(\alpha) = E\left(u''\left(w + \alpha(X - p)\right)(X - p)^2\right)$$

and the fact that $(X - p)^2 \geq 0$ and $u'' \leq 0$. Note: we just moved the derivative inside the expectation. You can do this basically always.

The second is due to $f'(0) = E\left(u'\left(w\right)(X - p)\right) = u'(w)E(X - p) > 0$ due to $E(X) > p$ and $u' > 0$.

Now we show the following:

Lemma 3.1. For this maximization problem, $\alpha^* > 0$.

Proof. First, $f(\alpha)$ is continuous so $\alpha^* \in [0, 1]$ exists. Since $f'(0) > 0$, it cannot be that $\alpha^* = 0$, since the necessary conditions for optimality would then imply that $f'(0) \leq 0$ (see previous notes). Thus $\alpha^* > 0$. □

Note that concavity is not used for this.

Since f is concave, $\alpha^* \in (0,1)$ can be characterized through the FOC. Note: we know that $\alpha^* > 0$ but not that $\alpha^* \neq 1$. The FOC is $f'(\alpha) = 0$. Equivalently:

$$E\left(u'\left(w + \alpha(X-p)\right)(X-p)\right) = 0$$

4 Insuring Against Losses

Consider a random variable $L \geq 0$ with $E(L) > 0$. We interpret L as monetary loss. Let p be the price for full coverage. We say that p is actuarially fair if $p = E(L)$.

Week 3

Faheem Gilani and Hilal Taşkan

Spring 2025

Topics

1. States and Random Variables
   (a) Representation
   (b) Concavity and Diversification
2. Savings for the Future
   (a) Time Discounting
   (b) Probability of Surviving to the next period
   (c) Value function for Time Discounting
   (d) Paying for a Medical Procedure.

1 States and Random Variables

1.1 State Space Representation of Random Variables

A finite random variable X is a mapping from a set of N states $\Omega = \{\omega_1, \ldots, \omega_N\}$ to N outcomes $\{x_1, \ldots, x_N\}$ such that $X(\omega_i) = x_i$. The state ω_i occurs with probability p_i. We assume that $\sum_{i=1}^{N} p_i = 1$ and $p_i > 0$ for all i. The story is that the state ω_i is realized with probability p_i and the random variable X takes on the value x_i when state ω_i is realized. In general, finite random variable can be thought of as points in N space.

For example, let $\Omega = \{\mathrm{G}, \mathrm{B}\}$. Assume that the probability of $\mathrm{G} = 1/2$. Let F_1 be the random variable that takes on the value 1.6 in state G and .8 in state B and let F_2 be the random variable that takes on the value .8 in state G and 1.6 in state B. Then F_1 can be represented as the point $(1.6, .8)$ in $2-D$, with the x- axis labeled as "G" and the y-axis labeled as "B". Similarly, F_2 would be the point $(.8, 1.6)$.

1.2 Concavity and Diversification

A risk averse investor with wealth w faces random variables F_1 and F_2 and wishes to invest an amount α in F_1 and the remaining $(1-\alpha)$ in F_2. Their expected utility from this is

$$E\left(u\left(w + \alpha F_1 + (1-\alpha)F_2\right)\right) = \frac{1}{2}u\left(w + 1.6\alpha + .8(1-\alpha)\right) + \frac{1}{2}u\left(w + .8\alpha + 1.6(1-\alpha)\right).$$

Taking the derivative of the preceding in α and setting it equal to 0 yields $\alpha = 1/2$. This means that the investor allocated half their resources into F_1 and half into F_2. They therefore face $\frac{1}{2}F_1 + \frac{1}{2}F_2$. Notice that this random variable takes on the value 1.2 in either state and therefore has 0 mean. Thus, by diversifying (choosing 1/2 each in this case), the decision maker was able to lower their risk (the variance of returns) to 0.

This idea of diversification is seen everywhere. For example, in poor, farming based villages, it is common for families that experience counter cyclical crop yields (that is, one family has high yield when the other has low yield and vice-versa) to join through marriage. The idea being that these families now experience lower variance (risk) in yield.

2 Savings for the Future

Consider the maximization problem

$$V(w) = \max_{x \geq 0} \left(u(w-x) + u(x)\right),$$

where u is strictly concave. This is a model of savings decision for a risk averse decision maker who lives for two periods: they have an initial wealth w and wish to save an amount x of their wealth (and consumer the remaining this period) and then consumer the amount they saved in the next period.

Notice that the objective $f(x, w) = u(w - x) + u(x)$ is supermodular due to $f_{xw} = -u''(w - x) > 0$ so by Topkis, $x^*(w)$ is increasing. Thus an individual with more initial wealth will save more for their future.

2.1 Time discounting

Consider instead the problem

$$V(w, \beta) = \max_{x \geq 0} \left(u(w - x) + \beta u(x) \right),$$

where $0 < \beta < 1$. This is similar to the model above except now the decision maker discounts the future. Note now that $f(x, \beta, w)$ is supermodular in β and x so long as u is increasing. Indeed, $f_{x\beta} = u'(x) > 0$. So $x^*(\beta, w)$ is increasing in β (holding w fixed). Note that this result does not depend on the concavity of u. Consequently, more patient decision makers (those with higher β, equivalently, those that discount the future less), will save more for the future.

2.2 A different interpretation of discounting

Here is a different interpretation of the preceding model. Suppose that the decision maker is alive today but has a probability p of living to the next period. Then they are maximizing

$$\max_{x \geq 0} \left(u(w - x) + pu(x) + (1 - p) \cdot 0 \right) = \max_{x \geq 0} \left(u(w - x) + pu(x) \right).$$

The objective here consists of today's utility $u(w - x)$ and the expected utility of consuming $u(x)$ in the next period. Since the decision maker lives to the next period with probability p, their expected utility in the next period will be $pu(x)$ (they will get utility $u(x)$ with probability p and die and therefore get utility 0 with probability $1 - p$). Under this interpretation, optimal savings increasing in the probability of surviving to the next period.

2.3 Value Function for the Time Discounting Problem

Consider again

$$V(p) = \max_{x \geq 0} \left(u(w - x) + pu(x) \right).$$

Note that w is also a parameter of this problem but we are omitting it from the value function because we only care about the change in value with respect to p. By the Envelope theorem,

$$V'(p) = u(x^*).$$

The latter is positive so long as $u \geq 0$, $u' > 0$ and and $x^* > 0$. So the value function is increasing in the probability of survival into the future (equivalently, an increase in the time discounting factor).

2.4 Paying for a Medical Procedure

Suppose the decision maker can pay an amount c in the current period to increase p to q. The maximization problem is

$$V_1(w - c, c) = \max_{x \geq 0} u(w - c - x) + qu(x)$$

The original maximization problem is

$$V_2(w, p) = \max_{x \geq 0} u(w - x) + pu(x)$$

Notice that V_1 is increasing in q and decreasing in c. So its possible to have a value $c^*(q)$ such that $V_1(w - c^*(q), q) = V_2(w, p)$.

Week 4

Faheem Gilani and Hilal Taşkan

Spring 2025

Topics

1. Partnerships
2. Fishing Fleets
3. Formal Definition of Games
4. Rational Pigs
5. Matching Pennies

1 Partnerships

Consider the two players with utility

$$u_1(x,y) = \frac{1}{2}\Pi(x,y) - c_1(x)$$
$$u_2(x,y) = \frac{1}{2}\Pi(x,y) - c_2(y).$$

Here player 1 chooses x and player 2 chooses y. Suppose that

$$\frac{\partial^2 \Pi(x,y)}{\partial x \partial y} \geq 0.$$

This is a model of a firm whose profits $\Pi(x,y)$ depend on the efforts of two partners and are equally shared among them. Here x is the effort of partner 1 and y the effort of partner 2. The cross partial assumption implies that marginal profits with respect to partner i's effort is higher at higher levels of partner j's efforts.

The effort pair (x^*, y^*) is a Nash equilibrium of this game if

$$x^* \in \arg\max_{x \geq 0} u_1(x, y^*)$$
$$y^* \in \arg\max_{y \geq 0} u_1(x^*, y).$$

The first order conditions for the Nash equilibria are

$$\frac{1}{2}\frac{\partial \Pi(x^*,y^*)}{\partial x} - c_1'(x^*) = 0$$
$$\frac{1}{2}\frac{\partial \Pi(x^*,y^*)}{\partial x} - c_2'(y^*) = 0.$$

These are sufficient when the objectives are concave in own effort, which we will assume here.

We say that (x,y) is **efficient** if there is no (x',y') and player i such that $u_i(x',y') > u_i(x,y)$ and $u_j(x',y') \geq u_j(x,y)$ for $j \neq i$. In other words, no player can be made better off without making some player worse off. We show that the Nash equilibria is not efficient by showin that is does not solve

$$\max_{x,y \geq 0} \Pi(x,y) - c_1(x) - c_2(y).$$

The objective function here is the sum of the two player's utility and the fact that the Nash does not solve this implies that the sum of player's utilities can be strictly improved, implying that (x^*, y^*) is not efficient. The first order conditions for this problem are

$$\Pi(x,y) - c_1'(x) = 0$$
$$\Pi(x,y) - c_2'(y) = 0.$$

These conditions are sufficient when the objective is concave and in that case we can see that the Nash solution cannot coincide with these FOCs. So the Nash equilibrium is not efficient.

2 Fishing Fleets

Consider I players, indexed $1, 2, \ldots, I$ with utilities

$$u_i(q_i, q_{-i}) = q_i V(Q) - cq_i,$$

where $q = (q_1, \ldots, q_I)$, $q_i \geq 0$ for all i, $q_{-i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_I)$ and $Q = \sum_{i=1}^{I} q_i$.

The interpretation is the following. There are I firms and Each firm is choosing a fleet size $q_i \geq 0$ to send to the ocean for fishing. The total catch depends inversely on the total fleet size of all firms on the ocean and fleets have marginal cost c of operation.

In this case, $q^* = (q_1^*, \ldots, q_I^*)$ is a Nash equilibria of this game if for all $i \in \{1, \ldots, I\}$,

$$q_i^* \in \arg\max_{q_i \geq 0} u_i(q_i, q_{-i}).$$

If the utilities are concave in own choices then the FOCs are sufficient. The FOCs are:

$$V(Q) + q_i V'(Q) = c.$$

Adding all of these equations and dividing by I, we obtain

$$V(Q) + \frac{1}{I} V'(Q) = c.$$

Notice that this approaches $V(Q) = c$ when $I \uparrow \infty$. So in the limit, utilities are 0. So in the limit we approach the "most inefficient outcome possible".

For more detail: See Homework 2.

3 Formal Definition of Games

Let $I = \{1, \ldots, I\}$ be a finite index set. Notice the abuse of notation: the maximum in this set is I and we also name this set I. For each $i \in I$ let A_i be a set. This set can be anything. Let $A = A_1 \times A_2 \times \ldots \times A_I$ be the set of all ordered pairs $a = (a_1, \ldots, a_I)$, where $a_i \in A_i$.

	u_1	u_2
(Push, Push)	-4	6
(Push, Wait)	-4	10
(Wait, Push)	5	1
(Wait, Wait)	0	0

Table 1: Utility Specification for player Rational Pigs

	Push	Wait
Push	(-4,6)	(-4,10)
Wait	(5,1)	(0,0).

Table 2: 2×2 representation of Rational Pigs

A game is a tuple $\Gamma = (A_i, u_i)_{i \in I}$ where $u_i : A \to \mathbb{R}$ is player i's utility as a function of all player's actions $a \in A$.

If $b_i \in A_i$ then the notation $a \backslash b_i$ is defined as

$$a \backslash b_i = (a_1 \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_I).$$

We say that $a^* \in A$ is a **Nash Equilibria** if for all $i \in I$,

$$a_i^* \in \arg\max_{b_i \in A_i} u_i(a^* \backslash b_i).$$

Rational Pigs

Suppose $I = \{1, 2\}$, $A_1 = \{\text{Push}, \text{Wait}\}$ and $A_2 = \{\text{Push}, \text{Wait}\}$. In order to define utility, we define it for all 4 possible ordered pairs that can arise. See Table 1.

Its common to represent 2 player 2 action games like these in a 2×2 table such as the one in Table 2 Notice that in this game, Wait strictly dominates wait for player 1, from which we find that the unique Nash equilibria of this game is (Wait,Push).

3.1 Matching Pennies

Consider the game shown in Table 3.

	H	T
H	(500,-500)	(-500,500)
T	(-500,500)	(500,-500).

Table 3: 2×2 representation of Rational Pigs

Week 5, Tuesday

Faheem Gilani

Spring 2025

Topics

1. Randomized (aka mixed) Equilibria.
2. Monitoring Games
3. Repeated Games and Grim Trigger

	L	R
U	$(9,4) \rightarrow$	$(3,6) \downarrow$
D	$(0,22) \uparrow$	$(5,9) \leftarrow$

Figure 1: A game that has no pure strategy Nash. Arrows represent best responses. The up/down arrows are best responses for player 1 and the left/right arrows are best reponses for player 2.

	L	R
U	$(10,10) \leftarrow$	$(-3,2) \downarrow$
D	$(-3,2) \uparrow$	$(4,4) \rightarrow$

Figure 2: A game that has both pure and mixed strategy Equilibria. Arrows represent best responses. The up/down arrows are best responses for player 1 and the left/right arrows are best responses for player 2.

1 Randomized Equilibria

The game in Figure 1 has no pure strategy equilibria (see the best response arrows). Nash's existence result guarantees that finite games have a Nash equilibria, but the equilibria may be mixed. To find the mixed strategy Equilibria for this game, we use the following method. Let β be the probability that player 1 plays U and let α be the probability that player 2 plays L. If $\beta > 0$ then player 1 gets the same expected utility from playing U and D. That is:

$$E(u_1(U,\beta)) = E(u_1(D,\beta))$$
$$\Longleftrightarrow$$
$$9\beta + 3(1-\beta) = 5(1-\beta).$$

This simplifies to $11\beta = 2$ so $\beta = 2/11$.

Similarly, if $\alpha > 0$ then player 2 gets the same expected utility from L and R. This yields the equation

$$4\alpha + 22(1-\alpha) = 6\alpha + 9(1-\alpha),$$

from which we find that $\alpha = 13/15$. So in a mixed strategy, player 1 plays U with probability $13/15$ and D with probability $2/15$ and player 2 plays L with probability $2/11$ and R with probability $9/11$.

The game in Figure 2 has two pure strategy equilibria: (U,L) and (D,R). Generally, finite games have an odd number of equilibria so we suspect there is a mixed strategy equilibria. To do this, assume again that player 1 plays U with probability α and player 2 plays D with probability β. We can use the indifference method from the previous paragraph to solve for the mixed strategy Nash equilibria in this game. Player 1's indifference condition is:

$$10\beta - 3(1-\beta) = -3\beta + 4(1-\beta),$$

	Monitor	Other Work
Cheat	$(0, B) \downarrow$	$(w + c, b) \leftarrow$
Rules	$(w, 0) \rightarrow$	$(w, b) \uparrow$

Figure 3: A monitoring game. There are no Pure Strategy Nash. Here $B > b > 0$

from which we find that $\beta = 7/20$ and Player 2's indifference condition is

$$10\alpha + 2(1 - \alpha) = 2\alpha + 4(1 - \alpha),$$

from which we find that $\alpha = 1/5$.

2 Monitoring Games

Consider the game in Figure 3. This is a game between a monitor (player 2) and a worker (player 1). The monitor can choose between monitoring (auditing) the worker to make sure they are working or doing other things. The worker can cheat (not work) or follow the rules (work). The monitor has a net reward B for catching a cheating worker and a fixed reward $b < B$ from doing other work. The worker obtains a base pay w so long as they are working or so long as they are not caught. Work costs the worker c. To find the mixed strategy equilibria in this game, we suppose the player 1 plays Cheat with probability α and player 2 plays Monitor with probability β. Player 1's indifference condition is

$$(w + c)(1 - \beta) = w,$$

from which we find that $\beta = 1 - \frac{w}{w+c} = \frac{c}{w+c}$. Player 2's indifference condition is

$$\alpha B = b,$$

from which we find that $\alpha = \frac{b}{B}$. In this problem, α is the probability that player 1 cheats. To make this small, we need either b (the utility to the monitor from other work) to be small or B (the net reward for catching a cheater) to be large.

3 Repeated Games

	Cheat	Honest
Cheat	$(0, 0)$	$(10, -5)$
Honest	$(-4, 13)$	$(7, 9)$

Figure 4: A game between between two parties that do not trust each other.

Consider a repeated interaction of the game described in Figure 4. Players meet at time $t = 0, 1, 2, 3, \ldots$ and each t they play the game in this table (they decide between cheating

the other player and taking an honest action). Let u_t^i be player i's utility in period t. We assume that player i's lifetime utility from the interaction is $\sum_{t=0}^{\infty} \beta^t u_t^i$, where $\beta \in (0,1)$.

A grim trigger strategy for both players in this game is the following: both players play the honest action on day 0. Then at $t > 0$, both players are prescribed to play the honest action so long as all players have played the honest action in the past (at all times $s < t$). If either player has cheated in the past at some point, then both players are prescribed to cheat. If both players follow this strategy then player 1 obtains a lifetime utility of

$$7 + 7\beta + 7\beta^2 + \ldots = 7\sum_{t=0}^{\infty} \beta^t = \frac{7}{1-\beta},$$

where the last equality follows from the formula for an infinite geometric series: $\sum_{t=0}^{\infty} \beta^t = (1-\beta)^{-1}$ when $\beta \in (0,1)$. If instead player 1 decides to cheat at time T while the other player follows the grim trigger strategy, player i obtains utility

$$7 + 7\beta + \ldots 7\beta^{T-1} + 10\beta^T + 0\beta^{T+1} + \ldots = 7\sum_{t=0}^{T-1} \beta^t + 10\beta^T = \frac{1-\beta^T}{1-\beta} + 10\beta^T,$$

where we have used the formula for a finite geometric series: $\sum_{t=0}^{T-1} \beta^t = \frac{1-\beta^T}{1-\beta}$. Thus, grim trigger will be obeyed so long as

$$\frac{7}{1-\beta} \geq \frac{1-\beta^T}{1-\beta} + 10\beta^T.$$

This puts a restriction on β for which grim trigger can be supported as a Nash equilibria in the repeated game for player 1. We would need to do this exercise for player 2 as well in order to find the range of β for which grim trigger can be supported.

02.20.2025, Thursday

Faheem Gilani, Hilal Taşkan

Spring 2025

Topics

1. Review
2. Concave Optimization Problems: Interior and Boundary Solutions
3. Analyzing Shifts in Net Marginal Benefit
4. Value Functions and Envelope Theorem
5. Externalities in the Game, Sub-game Perfection, Dominant Strategies

1 Derivatives

- $f'(x) = \frac{f(x+h)-f(x)}{h}$, as $h \to 0$
- $\frac{\partial f(x,y)}{\partial x} = \frac{f(x+h,y)-f(x,y)}{h}$, as $h \to 0$ (holding y fixed)

2 Increasing/Decreasing Net Marginal Benefits

3 approaches:

(a) Diagram

(b) Implicit differentiation of FOCs

(c) Supermodularity or Submodularity

3 Random Variables

- Let $P(X = x_i) = p_i$ for $i = 1, \ldots, n$
- $E[X] = \sum x_i p_i$
- $P(a \leq X \leq b) = \int_a^b f(x)dx$

- $E[X] = \int_{-\infty}^{+\infty} xf(x)dx$
- $Var(X) = E[(X - E[X])^2]$, $EV(X) = Var(X)$
- $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
- $EV(X + Y) = Cov(X, Y)$

4 Tangent Line and Optimization

- Linear approximation or tangent line at x_0 is given by:

$$T_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$$

- Key Insight: If $f'(x_0) < 0$, then $f(x) < T_{x_0}(x)$. The function always remains below its tangent line, except at the point x_0.
- Specifically, if $f' = 0$, the tangent line is horizontal, indicating a local maximum since the function does not exceed its tangent.
- In calculus, a maximum occurs when the derivative equals zero. For concave functions, a zero derivative implies the presence of a maximum.
- In game theory, equilibrium is found when all players reach a mutual optimum. If each player's payoff function is concave, the FOCs define the conditions for equilibrium, representing mutually best responses.

5 FOCs and Concavity

Consider the problem

$$\max_{x \in [a,b]} f(x).$$

Suppose f is continuously differentiable in x and let $x^* \in [a, b]$ be the optimizer (note that such an optimum exists due to the continuity of f). Then the following *necessary conditions* hold.

- If $x^* \in (a, b)$ then $f'(x^*) = 0$.
- If $x^* = a$ then $f'(a) \leq 0$.
- If $x^* = b$ then $f'(b) \geq 0$.

If f is concave then the following *sufficient conditions* also hold:

- If x solves $f'(x) = 0$ and $x \in [a, b]$, then $x = x^*$.
- If $f'(a) \leq 0$ then $a = x^*$.
- If $f'(b) \geq 0$ then $b = x^*$.

Consider the problem

$$\max_{x \geq a} f(x).$$

Suppose f is continuously differentiable in x and that $x^* \in [a, \infty)$ exist. Then the following *necessary conditions* hold.

- If $x^* > 0$ then $f'(x^*) = 0$.
- If $x^* = a$ then $f'(a) \leq 0$.

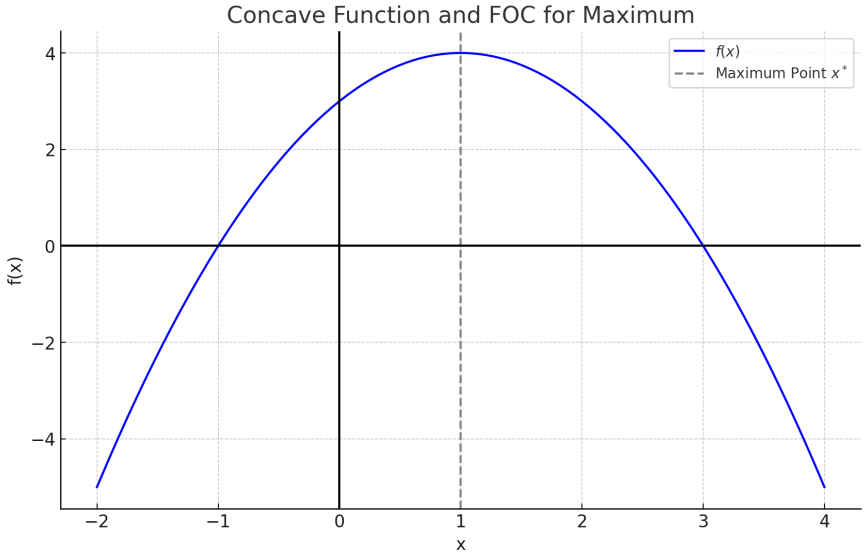
If f is concave then we additionally have the following *sufficient conditions*:

- If $f'(x) = 0$ and $x \geq 0$ then $x = x^*$.
- If $f'(0) \leq 0$ then $0 = x^*$.

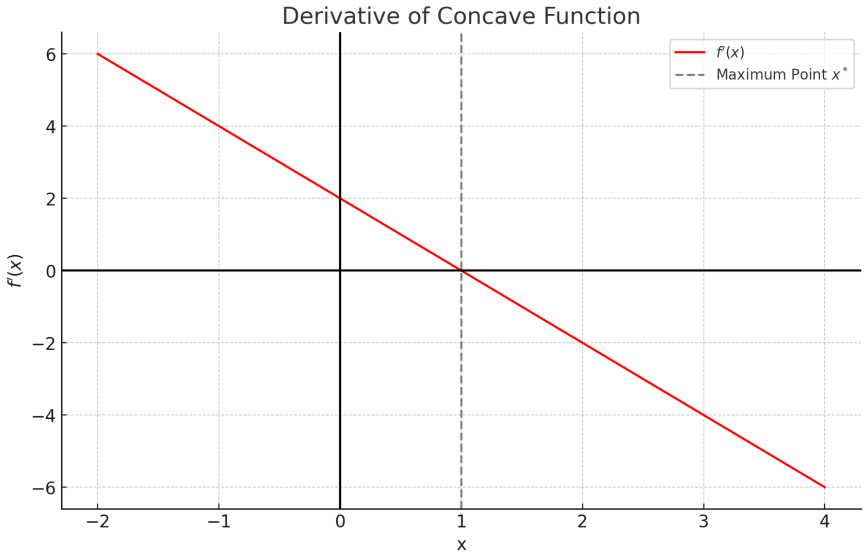
Graph Analysis:

- $f' > 0$ to the left of the maximum and $f' < 0$ to the right of the maximum.
- The first derivative (f') crosses zero only at the maximum point.
- To the left of this point, the slope is positive, and to the right, it is negative.

- Concave Function and FOC for Maximum:



- Derivative of Concave Function:



Let us consider maximizing the concave function in a bounded interval.

$$\max_{a \leq x \leq b} f(x)$$

We have 3 cases to consider:

- Interior Solution
- Solution at the lower bound
- Solution at the upper bound

Interior Solution

$$\max_{a \le x \le b} f(x) = -(x-3)^2 + 10$$

- Objective Function: $f(x) = -(x-3)^2 + 10$
- Domain: $a = 0,\ b = 5$

*Analysis

- $f(x)$ is **concave** because the second derivative is negative: $f''(x) = -2 < 0$.
- The first-order condition:

$$f'(x) = -2(x-3) = 0 \implies x = 3$$

- Since 3 is within the interval $[0, 5]$, the solution is **interior**.

*Optimal Solution

$$x^* = 3, \quad f(3) = 10$$

Solution at Lower Bound

$$\max_{a \le x \le b} f(x) = -(x-1)^2 + 5$$

- Objective Function: $f(x) = -(x-1)^2 + 5$
- Domain: $a = 2,\ b = 4$

*Analysis

- $f(x)$ is **concave** because the second derivative is negative: $f''(x) = -2 < 0$.
- The first-order condition:

$$f'(x) = -2(x-1) = 0 \implies x = 1$$

- Since $x = 1$ is **outside** the feasible region, check the boundaries:
  - $f(2) = -(2-1)^2 + 5 = 4$

- $f(4) = -(4-1)^2 + 5 = -4$

- The maximum occurs at the **lower bound**.

*Optimal Solution

$$x^* = 2, \quad f(2) = 4$$

Solution at Upper Bound

$$\max_{a \le x \le b} f(x) = -(x-4)^2 + 8$$

- Objective Function: $f(x) = -(x-4)^2 + 8$
- Domain: $a = 0,\ b = 3$

*Analysis

- $f(x)$ is **concave** because the second derivative is negative: $f''(x) = -2 < 0$.
- The first-order condition:

$$f'(x) = -2(x-4) = 0 \implies x = 4$$

- Since $x = 4$ is **outside** the feasible region, check the boundaries:
  - $f(0) = -(0-4)^2 + 8 = -8$
  - $f(3) = -(3-4)^2 + 8 = 7$
- The maximum occurs at the **upper bound**.

*Optimal Solution

$$x^* = 3, \quad f(3) = 7$$

Summary of Solutions

- **Interior Solution**: $x^* = 3$ within $[0, 5]$
- **Lower Bound Solution**: $x^* = 2$ at $a = 2$
- **Upper Bound Solution**: $x^* = 3$ at $b = 3$

6 Analyzing Shifts in Net Marginal Benefit

Implicit Differentiation and Net Marginal Benefit

Let's examine the problem using an **implicit differentiation approach**, assuming that the optimal solution is **interior** (i.e., not at the boundary):

$$V(\theta) = \max_x f(x, \theta)$$

Assume the following concavity condition:

$$\frac{\partial^2 f(x, \theta)}{\partial x^2} < 0 \quad \text{for all } \theta.$$

This implies that the problem is **strictly concave**, ensuring that any critical point is a global maximum.

If the second derivative is negative and the optimum does not occur at a boundary, then this critical point is indeed the maximum.

Optimality Condition

The optimal choice $x^*(\theta)$ satisfies the first-order condition:

$$\frac{\partial f(x^*(\theta), \theta)}{\partial x} = 0$$

We are now ready to investigate the value function using the **implicit differentiation approach** to understand supermodularity and submodularity, given the assumptions made above.

The objective is to find the derivative:

$$\frac{dx^*(\theta)}{d\theta}$$

More precisely, we want to determine the sign of the derivative:

$$\frac{dx^*(\theta)}{d\theta}$$

Starting from the first-order condition for optimality:

$$\frac{\partial f(x^*(\theta), \theta)}{\partial x} = 0$$

Using Implicit Differentiation

To find how the optimal choice changes with θ, we differentiate both sides with respect to θ. Note that θ influences the equation in two ways:

- **Indirectly**: Through $x^*(\theta)$, since the optimal choice depends on θ.
- **Directly**: As an explicit parameter in the function.

Applying the chain rule:

$$\frac{d}{d\theta}\left[\frac{\partial f(x^*(\theta), \theta)}{\partial x}\right] = \frac{\partial^2 f(x^*(\theta), \theta)}{\partial x^2}\frac{dx^*(\theta)}{d\theta} + \frac{\partial^2 f(x^*(\theta), \theta)}{\partial x \partial \theta} = 0$$

Since the problem is **strictly concave**, the second derivative is negative:

$$\frac{\partial^2 f}{\partial x^2} < 0$$

Solving for $\frac{dx^*(\theta)}{d\theta}$

Rearranging the equation:

$$\frac{dx^*(\theta)}{d\theta} = -\frac{\frac{\partial^2 f(.)}{\partial x \partial \theta}}{\frac{\partial^2 f}{\partial x^2}}$$

Given that $\frac{\partial^2 f}{\partial x^2} < 0$, the sign of $\frac{dx^*(\theta)}{d\theta}$ is determined by the sign of the **cross-partial derivative**:

$$\frac{dx^*(\theta)}{d\theta} > 0 \text{ if } \frac{\partial^2 f(.)}{\partial x \partial \theta} > 0$$

Therefore, the derivative $\frac{dx^*(\theta)}{d\theta}$ is **positive** if the **cross-partial derivative** is positive, indicating that an increase in θ leads to an increase in the optimal choice $x^*(\theta)$.

Supermodularity and Net Marginal Benefit

Discrete Version

A function $f : \mathbb{R}^2 \to \mathbb{R}$ is **supermodular** or has **increasing differences** in (x, θ) if, for all $x' > x$, the following holds:

$$f(x', \theta) - f(x, \theta) \text{ is nondecreasing in } \theta.$$

- If f is supermodular in (x, θ), then the incremental gain from choosing a higher x is greater when θ is higher.
- Supermodularity is equivalent to the property that if $\theta' > \theta$, then:

$$f(x, \theta') - f(x, \theta) \text{ is nondecreasing in } x.$$

Differentiable Version

When f is smooth, supermodularity can be characterized in terms of its derivatives.
A twice continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ is **supermodular** in (x, θ) if and only if:

$$\frac{\partial^2 f}{\partial x \partial \theta} \geq 0 \quad \text{for all } (x, \theta).$$

This condition can be interpreted as:

- The **cross-partial derivative** is nonnegative:

$$\frac{\partial^2 f}{\partial x \partial \theta} \geq 0$$

- The inequality in the definition of supermodularity corresponds to the discrete version of this cross-partial condition.

Supermodularity and Comparative Statics

Supermodularity is sufficient to draw **comparative statics conclusions** in optimization problems.

Topkis's Monotonicity Theorem

Theorem (Topkis' Theorem)

- If f is supermodular in (x, θ), then the optimal solution $x^*(\theta)$ is **non-decreasing** in θ.

Explanation

- Supermodularity in (x, θ) means that the **cross-partial derivative** is nonnegative:

$$\frac{\partial^2 f}{\partial x \partial \theta} \geq 0$$

- This condition implies that an increase in θ makes choosing a higher x more attractive, leading to a non-decreasing optimal choice:

$$\theta' > \theta \implies x^*(\theta') \geq x^*(\theta)$$

- This is a powerful result because it provides a monotonic relationship between the parameter θ and the optimal choice $x^*(\theta)$ without needing explicit solutions.

7 Envelope Theorem

Assume that the optimal choice $x^*(\theta)$ satisfies the first-order condition:

$$\frac{\partial f(x^*(\theta), \theta)}{\partial x} = 0$$

This implies that $x^*(\theta)$ maximizes the function. Consequently, the maximized value of the objective function is:

$$V(\theta) = f(x^*(\theta), \theta)$$

We aim to calculate $\frac{dV}{d\theta}$. The parameter θ influences the value function both directly and indirectly.

According to the **Envelope Theorem**, the indirect effect can be ignored:

$$\frac{dV}{d\theta} = \frac{\partial f}{\partial x}\frac{dx^*(\theta)}{d\theta} + \frac{\partial f}{\partial \theta}$$

Eliminating the Indirect Effect

The indirect effect is zero because:

- The optimal choice x^* is made where the slope is flat ($f' = 0$).
- Therefore, when $x^*(\theta)$ changes, the slope remains unchanged because it is flat at the optimum.

Conclusion

Thus, we get:

$$\frac{dV}{d\theta} = \frac{\partial f}{\partial \theta} \quad \text{(only the direct effect matters).}$$

Game Setup with Two Players

Consider a scenario with two players:

$$I = \{1, 2\}$$

Each player has the following set of possible actions:

$$A_1 = [0, \infty)$$

$$A_2 = [0, \infty)$$

The utility functions for the players are:

$$u_1(a_1, a_2)$$

$$u_2(a_1, a_2)$$

Externalities in the Game

The fact that u_1 depends on a_2 and u_2 depends on a_1 suggests the presence of **externalities**, where each player's utility is influenced by the other player's actions.

- Specifically, if:
$$\frac{\partial u_1(a_1, a_2)}{\partial a_2} \neq 0, \quad \frac{\partial u_2(a_1, a_2)}{\partial a_1} \neq 0,$$
this indicates an **external effect** because one player's action impacts the other player's utility.
- Moreover, if:
$$\frac{\partial u_1(a_1, a_2)}{\partial a_2} > 0, \quad \frac{\partial u_2(a_1, a_2)}{\partial a_1} > 0,$$
then the actions create a **positive externality**, meaning each player's action positively affects the other's utility.
- Conversely, if these derivatives are negative, it indicates a **negative externality**, where one player's action adversely affects the other's utility.

Efficiency and Externalities in Equilibrium

When **externalities** are present, we typically do not expect the equilibrium to be **efficient**.

Recall: Nash Equilibrium Definition

(a_1^*, a_2^*) is a Nash equilibrium if:

1. Player 1's choice a_1^* maximizes their utility, given Player 2's action:

$$\max_{a_1 \geq 0} u_1(a_1, a_2^*)$$

2. Player 2's choice a_2^* maximizes their utility, given Player 1's action:

$$\max_{a_2 \geq 0} u_2(a_1^*, a_2)$$

Explanation

- In (1), **Player 1** treats **Player 2's** action a_2^* as a **fixed value** or a **parameter**, meaning Player 1 takes it as given and cannot influence it.
- In (2), **Player 2** similarly treats **Player 1's** action a_1^* as fixed.

This is consistent with the concept of **simultaneous-move games**, where each player optimizes their utility, assuming the other player's action is fixed.

First-Order Conditions and Concavity

The equilibrium can be characterized using **First-Order Conditions (FOCs)** if the following hold:

- $u_1(a_1, a_2)$ is **concave** in a_1 for every fixed a_2
- $u_2(a_1, a_2)$ is **concave** in a_2 for every fixed a_1

Under these conditions, the **FOCs** for the Nash equilibrium are:

$$\frac{\partial u_1(a_1^*, a_2^*)}{\partial a_1} = 0$$

$$\frac{\partial u_2(a_1^*, a_2^*)}{\partial a_2} = 0$$

These conditions are based on the assumption that the FOCs fully describe the solution to each player's optimization problem.

Subgame Perfection

Assume that Player 2's best response is given by:

$$a_2^*(a_1) = \arg\max_{a_2} u_2(a_1, a_2)$$

Player 1 can **anticipate** Player 2's reaction to their choice. The key idea of **subgame perfection** is that Player 1 uses **backward induction**, looking ahead to how Player 2 will respond.

Therefore, Player 1 maximizes their utility by choosing a_1 while taking into account Player 2's best response:

$$\max_{a_1} u_1(a_1, a_2^*(a_1))$$

Here, Player 1 knows that Player 2 will **optimally respond** by selecting $a_2^*(a_1)$ based on Player 1's choice of a_1.

First-Order Conditions in Simultaneous-Move Game

For a **simultaneous-move game**, the **First-Order Conditions (FOCs)** are:

$$\frac{\partial u_1(a_1^*, a_2^*)}{\partial a_1} = 0$$

$$\frac{\partial u_2(a_1^*, a_2^*)}{\partial a_2} = 0$$

In this case, each player takes the other's action as given.

First-Order Condition in Subgame Perfection

In a **sequential-move game** with subgame perfection, the FOC for Player 1 becomes:

$$\frac{\partial u_1}{\partial a_1} + \frac{\partial u_1}{\partial a_2}\frac{da_2^*}{da_1} = 0$$

This expression shows that Player 1 not only considers the direct effect of their choice on their utility but also the **indirect effect** through Player 2's **optimal reaction**.

Key Difference

- The **FOCs** are different in simultaneous-move and sequential-move games.
- In **subgame perfection**, Player 1 accounts for the strategic influence of their choice on Player 2's best response, reflecting a forward-looking, strategic perspective.

Game Setup

- Set of players:
$$I = \{1, 2\}$$
- Action sets:
$$A_1 = \{U, D\} \quad \text{(Player 1: Up, Down)}$$
$$A_2 = \{L, R\} \quad \text{(Player 2: Left, Right)}$$
- There are **4 possible outcomes** and therefore **8 utility numbers** to describe all payoffs.

Game 1: Both Players Have Dominant Strategies

	L	R
U	$(2, 2)$	$(0, 3)$
D	$(3, 0)$	$(1, 1)$

Analyzing Dominant Strategies

Player 1's Strategy:

- If Player 2 plays L, Player 1's payoffs are:
$$U \to 2, \quad D \to 3$$
Player 1 prefers D if Player 2 plays L.
- If Player 2 plays R, Player 1's payoffs are:
$$U \to 0, \quad D \to 1$$
Player 1 prefers D if Player 2 plays R.
- **Conclusion**: D is a **dominant strategy** for Player 1.

Player 2's Strategy:

- If Player 1 plays U, Player 2's payoffs are:

$$L \to 2, \quad R \to 3$$

  Player 2 prefers R if Player 1 plays U.

- If Player 1 plays D, Player 2's payoffs are:

$$L \to 0, \quad R \to 1$$

  Player 2 prefers R if Player 1 plays D.

- **Conclusion**: R is a **dominant strategy** for Player 2.

Nash Equilibrium

The **Nash equilibrium** is:

$$(D, R) \text{ with payoffs } (1, 1)$$

Both players are playing their **dominant strategies**.

Game 2: No Dominant Strategies

	L	R
U	$(3, 2)$	$(1, 4)$
D	$(2, 3)$	$(4, 1)$

Analyzing Strategies

Player 1's Strategy:

- If Player 2 plays L, Player 1's payoffs are:

$$U \to 3, \quad D \to 2$$

  Player 1 prefers U if Player 2 plays L.

- If Player 2 plays R, Player 1's payoffs are:

$$U \to 1, \quad D \to 4$$

  Player 1 prefers D if Player 2 plays R.

- **Conclusion**: Player 1 **does not** have a **dominant strategy**.

Player 2's Strategy:

- If Player 1 plays U, Player 2's payoffs are:

$$L \to 2, \quad R \to 4$$

  Player 2 prefers R if Player 1 plays U.

- If Player 1 plays D, Player 2's payoffs are:

$$L \to 3, \quad R \to 1$$

  Player 2 prefers L if Player 1 plays D.

- **Conclusion**: Player 2 **does not** have a **dominant strategy**.

Week 6, Thursday

Faheem Gilani

Spring 2025

Topics

1. KKT Conditions and Examples
2. Production

1 KKT Conditions

Consider the problem

$$V(b) = \max_{\{x \in \mathbb{R}^n : g(x) \leq \mathbf{b}\}} f(x).$$

Here $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $b \in \mathbb{R}^m$. This is equivalent to writing:

$$\begin{cases} V(b) & = \max_{(x_1,\ldots,x_n)\in\mathbb{R}^n} f(x_1, \ldots, x_n) \\ st. & g_1(x_1, \ldots, x_n) \leq b_1 \\ & \vdots \\ & g_m(x_1, \ldots, x_m) \leq b_m. \end{cases}$$

The Lagrangian associated with this problem is

$$\begin{aligned} \mathcal{L}(x, \lambda) &= f(x) + \lambda \cdot (b - g(x)) \\ &= f(x_1, \ldots, x_n) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x_1, \ldots x_n)). \end{aligned}$$

Here, recall that if $u, v \in \mathbb{R}^k$ then the dot product between u and v is $u \cdot v = \sum_{i=1}^{k} u_i v_i$.

The Kuhn Tucker conditions are:

$$\begin{aligned} D_x \mathcal{L}(x, \lambda) &= \mathbf{0} \\ g(x) &\leq \mathbf{b} \\ \lambda \cdot (b - g(x)) &= \mathbf{0} \\ \lambda &\geq \mathbf{0}. \end{aligned}$$

You may translate these into its non-vector form for practice. The main theorem is the following.

Theorem 1.1.

- (Necessity) If f is continuously differentiable and if *Constraint qualification* holds then at any optimum $x^*(b)$ there exists some $\lambda \in \mathbb{R}^m$ such that the KKT conditions are satisfied.
- (Sufficiency) If f is furthermore concave in $(x_1, \ldots, x_n)$ and each g_i is convex in $(x_1, \ldots, x_n)$ and the KKT conditions are satisfied for some (x, λ) then $x^*(b) = x$. That is, x is an optimum.

We say that constraint $j \in \{1, \ldots, m\}$ is *binding* if $g_j(x^*) = b_j$. That is, at the optimum, the constraint holds with equality. We say that constraint j is *slack* otherwise. Note that if constraint j is slack then, since each $\lambda_i \geq 0$, its corresponding Lagrange multiplier λ_j must be 0 in order to satisfy the complementary slackness condition $\lambda_j(b_j - g_j(x)) = 0$.

A *nonnegativity constraint* is any constraint of the form $g_j(x) \geq 0$. The multiplies associated with nonnegativity constraints are often denoted by μ_j instead of λ_j.

The envelope theorem in this case is:

Theorem 1.2. Suppose (x^*, λ^*) solves the KKT conditions. Then, under certain conditions, $V'(b) = \partial_\lambda \mathcal{L}(x^*, \lambda^*) = \lambda^*$.

Example 1 Consider

$$\begin{aligned} &\max_{x \in \mathbb{R}} \sqrt{x} \\ st.\ &x \leq b \\ &x \geq 0, \end{aligned}$$

where $b > 0$.

This is in the general form described above with $n = 1$, $m = 2$, $f(x) = \sqrt{x}$, $g_1(x) = x$, $g_2(x) = -x$, $b_1 = b$, $b_2 = 0$. Here there is only one parameter: b. It is easy to see (since f is strictly increasing) that the maximizer is $x^*(b) = b$. So the value function is $V(b) = \sqrt{b}$. Its derivative is $V'(b) = \frac{1}{2\sqrt{b}}$. We solve this problem slowly, with way more detail than necessary.

The Lagrangian is

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda(b - x) + \mu x.$$

The KKT conditions are:

$$\begin{aligned} \frac{1}{2\sqrt{x}} - \lambda + \mu &= 0 \\ x &\leq b \\ x &\geq 0 \\ \lambda_1(b - x) &= 0 \\ \lambda_2 x &= 0 \\ \mu_1 &\geq 0 \\ \mu_2 &\geq 0. \end{aligned}$$

Since f is concave and the constraints are convex, these conditions are sufficient for characterizing x^*. To solve this system, we see that $x^*(b) \neq 0$ since, for example, $x = b/2$ yields

higher utility. The condition $\mu_2 x = 0$ then becomes $\mu_2 = 0$. This reduces the system to

$$\begin{aligned} \frac{1}{2\sqrt{x}} - \lambda &= 0 \\ x &\leq b \\ \mu_1(b - x) &= 0 \\ \mu_1 &\geq 0. \end{aligned}$$

Next, if $x < b$ then $\mu_1 = 0$ This would require

$$\frac{1}{2\sqrt{x}} = 0,$$

which is not possible. So it must be that $\lambda > 0$. This tells us that $x = b$. To check that $\lambda > 0$, we use the first equation to find that $\lambda = \frac{1}{2\sqrt{b}} > 0$ due to $b > 0$.

We could solve this faster by figuring out which constraints are binding and which are slack as follows. Since $\sqrt{x}$ is strictly increasing, $x^* = 0$ cannot be the optimum (in fact, we automatically know from this that $x^* = b$ but lets pretend we don't). So the constraint $x \geq 0$ is slack. Thus $\mu_2 = 0$. If $x < b$ (the first constraint is slack) then it must be that $\lambda = 0$ from which we would find that $\frac{1}{2\sqrt{x}} = 0$, which is not possible. So it must be that $x = 0$, from which we find that $\lambda^* = \frac{1}{2\sqrt{b}} \geq 0$.

The envelope theorem in this context tells us that $V'(b) = \lambda^* = \frac{1}{2\sqrt{b}}$

Example 2: Consider

$$\begin{cases} & \max_{(x_1,x_2)\in\mathbb{R}^2} x_1 x_2 \\ st. & 2x_1 + 2x_2 \leq L \\ & x_1 \geq 0 \\ & x_2 \geq 0, \end{cases}$$

Where $L > 0$.

The Lagrangian is $\mathcal{L}(x_1, \lambda, \mu_1, \mu_2) = x_1 x_2 + \lambda(L - 2x_1 - 2x_2) + \mu_1 x_1 + \mu_2 x_2$. The KKT conditions are

$$\begin{aligned} x_2 - 2\lambda + \mu_1 &= 0 \\ x_1 - 2\lambda + \mu_2 &= 0 \\ 2x_1 + 2x_2 &\leq L \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ \lambda(L - 2x_1 - 2x_2) &= 0 \\ \mu_1 x_1 &= 0 \\ \mu_2 x_2 &= 0 \\ \mu_1, \mu_2, \lambda &\geq 0. \end{aligned}$$

Again, these characterize the optimum because the objective is concave and the constraints are convex.

We solve these faster than in the preceding example as follows. If either $x_1 = 0$ or $x_2 = 0$, $f(x_1, x_2) = 0$ and this can be improved by setting $x_1, x > 0$. So neither of the non-negativity constraints are binding. Thus $\mu_1, \mu_2 = 0$. If $2x_1 + 2x_2 < L$ then $\lambda = 0$ and we would now find that $x_1, x_2 = 0$, which we argued is not the optimum. So it must be that $2x_1 + 2x_2 = L$. The first two constraints now imply that $x_1 = x_2 := x$. Substituting back we find that $4x = L$, from which $(x_1, x_2) = (L/4, L/4)$. Solving for λ we find that $\lambda = L/8 > 0$.

2 Production

Consider

$$\begin{aligned} V(X) = \max_{x_1, x_2} & f_1(x_1) + f_2(x_2) \\ st.\ & x_1 + x_2 \leq X \\ & x_1, x_2 \geq 0. \end{aligned}$$

Here $X > 0$. This is a general form of a constraint optimization problem that a central planner would solve in order to coordinate production. There are two firms and f_1 and f_2 are the production functions of firms 1 and 2, respectively. These firms use the same resource type to produce a good. There is a total amount of X of the resource available. The central planner wishes to allocate this total resource amongst the two firms in order to produce the most amount of the good possible.

You should write the Lagrangian yourself for practice. The KKT conditions in this case are

$$\begin{aligned} f_1'(x_1) - \lambda + \mu_1 &= 0 \\ f_2'(x_2) - \lambda + \mu_2 &= 0 \\ x_1 + x_2 &\leq X \\ x_1, x_2 &\geq 0 \\ \lambda(X - x_1 - x_2) &= 0 \\ \mu_1 x_1 &= 0 \\ \mu_2 x_2 &= 0. \end{aligned}$$

Note that the first two conditions tell us that at an optimum where both firms are producing (so when $\mu_1 = 0 = \mu_2$), it must be that $f'(x_1) = f'(x_2)$. That is, marginal product must be equalized amongst the two firms. In this situation, $V'(X) = \lambda^* = f'(x_1^*) = f_2'(x_2^*)$.

In the preceding, the central planner knew f_1 and f_2 and coordinated production in an optimal way. The result was efficient. We can achieve the same result by introducing prices.

Suppose the resource X costs a price p. Then each firm solves

$$\max_x f_i(x) - px$$
$$x \leq X.$$

Since the objective is concave, interior optima satisfy $f_1'(x) = p$ and $f_2'(x) = p$. So again $f_1'(x) = f_2'(x)$. To achieve the same solution as in the planner problem, we can use $p = \lambda^*$ (then the FOCs for interior optima will coincice).

Week 6, Thursday

Faheem Gilani

Spring 2025

Topics

1. Cobb Douglas: Quiz Solution
2. Equalizing Marginal Utility
3. Monotonic transformations
4. Inflation measures

1 Cobb Douglas: Quiz Solution

Consider the problem

$$\begin{aligned} V(p_1, p_2, w) = \max_{x_1, x_2} & x_1^\alpha x_2^\beta \\ st. \ & p_1 x_1 + p_2 x_2 \leq w \\ & x_1, x_2 \geq 0. \end{aligned}$$

The gradient of the objective function is

$$\left(\alpha x_1^{\alpha-1} x_2^\beta, \beta x_1^\alpha x_2^{\beta-1}\right).$$

Here $w > 0$and $p_1, p_2 \geq 0$.

This is the maximization problem associated with a consumer who has a budget and chooses between the amount of consumption of two goods with given market prices. The consumer has Cobb Douglas utility: $u(x_1, x_2) = x_1^\alpha x_2^\beta$.

The Lagrangian is:

$$\mathcal{L}(x_1, x_2, \lambda, \mu_1, \mu_2) = x_1^\alpha x_2^\beta + \lambda(w - p_1 x_1 - p_2 x_2) + \mu_1 x_1 + \mu_2 x_2.$$

The KKT conditions in non-vector form are:

$$\begin{aligned} \alpha x_1^{\alpha-1} x_2^\beta - \lambda p_1 + \mu_1 &= 0 \\ \beta x_1^\alpha x_2^{\beta-1} - \lambda p_2 + \mu_2 &= 0 \\ p_1 x_1 + p_2 x_1 &\leq w \\ x_1, x_2 &\geq 0 \\ \lambda(w - p_1 x_1 - p_2 x_2) &= 0 \\ \mu_1 x_1 &= 0 \\ \mu_2 x_2 &= 0 \\ \lambda, \mu_1, \mu_2 &\geq 0. \end{aligned}$$

Solving the KKT conditions

We assume that the KKT conditions are sufficient (this is the case here). So the maximizer $x^* = (x_1^*, x_2^*)$ is given by solving the KKT conditions. We solve as follows.

Note that $(x_1^*, x_2^*) > \mathbf{0}$ Indeed, we can improve upon $(0,0)$ by increasing each component: $u(0,y) = u(x,0) = 0 < u(a,b)$ for any $a, b > 0$ so we can improve the utility by giving the consumer some positive amount of each good, making sure that the amount we give is in their budget set (work this out for yourself for practice). Thus $\mu_1^*, \mu_2^* = 0$. Next, we argue that $\lambda^* > 0$. If not then the final condition implies that $\lambda^* = 0$. The first two conditions would then imply that $x_1^* = 0 = x_2^*$, which we argued is not the case. So it must be that $\lambda^* > 0$. From the complementary slackness for the budget constraint, we find that $p_1 x_1^* + p_2 x_2^* = w$. The conditions then reduce to:

$$\begin{aligned} \alpha x_1^{\alpha-1} x_2^{\beta} - \lambda p_1 &= 0 \\ \beta x_1^{\alpha} x_2^{\beta-1} - \lambda p_2 &= 0 \\ p_1 x_1 + p_2 x_2 &= 0 \\ \lambda &> 0. \end{aligned}$$

Since $\lambda, x_1, x_2 > 0$, we can combine the first two in order to obtain:

$$\begin{aligned} \frac{\alpha x_2}{\beta x_1} &= \frac{p_1}{p_2} \\ p_1 x_1 + p_2 x_2 &= w. \end{aligned}$$

We can now solve these by substitution. From the first equation we have $x_1 = \frac{\alpha p_2}{\beta p_1} x_2$. Substituting this into the second equation yields

$$\alpha p_2 x_2 + \beta p_2 x_2 = \beta w,$$

from which

$$x_2^* = \left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{w}{p_2}\right).$$

Substituting this back into either of the relations we found yields

$$x_1^* = \left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{w}{p_1}\right).$$

We can obtain the following indirect utility function (value function) by substituting this into the utility function:

$$\begin{aligned} V(p_1, p_2, w) &= \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{w}{p_1}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}\left(\frac{w}{p_2}\right)^{\beta} \\ &:= \kappa p_1^{-\alpha} p_2^{-\beta} w^{\alpha+\beta}, \end{aligned}$$

where

$$\kappa = \left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}$$

Ideal Inflation Measure

Suppose there are two goods, a consumer with Cobb Douglas utiltiy (as described in the preceding) and two time periods: $t = 0$, $t = 1$. The prices for the two goods in time 0 is $p^0 = (p_1^0, p_2^0)$ and the prices for the two goods in time 1 are (p_1^1, p_2^1). The consumer has wealth w_0 in period 0 and w_1 in period 1. In period 1, the consumer's indirect utility is

$$V_1(p_1^0, p_2^0, w_0) = \kappa (p_1^0)^{-\alpha} (p_2^0)^{-\beta} w_0^{\alpha+\beta}.$$

and the consumer's indirect utility in period 1 is

$$V_1(p_1^1, p_2^1, w_0) = \kappa (p_1^1)^{-\alpha} (p_2^1)^{-\beta} w_1^{\alpha+\beta}.$$

Suppose we wanted to know what value of w_1 would give the consumer the same indirect utility in period 1 as they had in period 0 under p_1^0, p_2^0, w_0. Then equating the preceding two indirect utilities and solving for w_1 yields

$$w_1 = \left((p_1^0/p_1^1)^{-\alpha} (p_2^0/p_2^1)^{-\beta}\right)^{\frac{1}{\alpha+\beta}} w_0.$$

Notice that κ cancels since it is independent of the time period.

Note that this expression is of the form $w_1 = \xi w_0$ where $\xi = \left((p_1^0/p_1^1)^{-\alpha} (p_2^0/p_2^1)^{-\beta}\right)^{\frac{1}{\alpha+\beta}}$. This ξ is the ideal *inflation measure* in this situation.

General Setup

We now consider the problem

$$\begin{aligned} V(p_1, p_2, w) = \max_{x_1, x_2 \geq 0} & \; u(x_1, x_2) \\ st. \; & p_1 x_1 + p_2 x_2 \leq w \\ & x_1, x_2 \geq 0, \end{aligned}$$

where u is assumed to be increasing (have positive gradient). The two constraints describe a triangle in $\mathbb{R}^2_+$ determined by the origin and the points $(0, w/p_2)$ and $(w/p_1, 0)$. The function $u(x, y)$ has isoclines (aka indifference curves) given by $I(u) = \{(x, y) \mid u(x, y) = u\}$. We assume that these are "bowed towards the origin". In this setup, the optimal solution is when the indifference curves are tangent to the outer boundary of the budget constraint (the line determined by $(0, w/p_2)$ and $(w/p_1, 0)$.

We show that the preceding is true via the KKT conditions. First, the fact that $Du >> 0$ means that $x^* >> \mathbf{0}$. This will imply that the multipliers for the non negativity constraints

are 0 so that we can ignore them. Next, the first two KKT conditions can be written as

$$\partial_{x_1} u(x_1, x_2) = \lambda p_1$$
$$\partial_{x_2} u(x_1, x_2) = \lambda p_2.$$

Week 6, Thursday

Faheem Gilani

Spring 2025

Topics

1. Substitutes
2. Quasilinear
3. Geometry of FOCs

1 Substitutes

Consider the problem

$$\begin{aligned} V(p,w) = \max_{x_1,x_2} \; & b_1x_1 + b_2x_2 \\ & p_1x_1 + p_2x_2 \leq w \\ & x_1, x_2 \geq 0. \end{aligned}$$

The Lagrangian is

$$\mathcal{L}(x,p,\lambda,\mu) = b_1x_1 + b_2x_2 + \lambda(w - p_1x_1 - p_2x_2) + \mu_1x_1 + \mu_2x_2.$$

The KKT conditions are:

$$\begin{aligned} b_1 - p_1\lambda + \mu_1 &= 0 \\ b_2 - p_2\lambda + \mu_2 &= 0 \\ p_1x_1 + p_2x_2 &\leq w \\ x_1, x_2 &\geq 0 \\ \lambda(w - p_1x_1 - p_2x_2) &= 0 \\ \mu_1x_1 &= 0 \\ \mu_2x_2 &= 0 \\ \lambda, \mu_1, \mu_2 &\geq 0. \end{aligned}$$

Before solving the equations, we can think about the łq"bang for your buck" principle to obtain a solution. In this case, the "utils per dollar" for good i is

$$\frac{\partial_{x_i} u(x_1, x_2)}{p_i} = \frac{b_i}{p_i}.$$

When $\frac{b_1}{p_1} > \frac{b_2}{p_2}$, the decisionmaker prefers to buy only of good 1. To understand this geometrically, notice that after rearranging we find that $\frac{b_1}{b_2} > \frac{p_1}{p_2}$. Notice that $u(x_1, x_2) = b_1x_1 + b_2x_2$ has linear indifference curves: $x_2 = \frac{k}{b_2} - \frac{b_1}{b_2}x_1$ with slope $\frac{b_1}{b_2}$. Recall that the slope of the budget line is $-\frac{p_1}{p_2}$. So in this case the slope of the indifference curve is steeper than the slope of the budget line, so the point of tangency is on the x_1 axis (the intercept point is $(w/p_1, 0)$).

A similar analysis with $\frac{b_1}{p_1} < \frac{b_2}{p_2}$ would yield the optimum $(0, w/p_2)$. Finally, when $\frac{b_1}{p_1} = \frac{b_2}{p_2}$, the analysis would yield that the consumer is indifferent between any combination of the bundles that lie on the budget line.

To solve this using the KKT conditions, we split it into cases.

$\mathbf{x_1^* > 0 \text{ and } x_2^* = 0}$. In this case, $\mu_1^* = 0$. The first two conditions then become:

$$\begin{aligned} b_1 - p_1\lambda &= 0 \\ b_2 - p_2\lambda + \mu_2 &= 0 \end{aligned}$$

Since $p_1 > 0$, substituting the first equation into the second yields

$$b_2 - \frac{p_2}{p_1}b_1 + \mu_2 = 0.$$

. Using $\mu_2 \geq 0$, the preceding becomes

$$b_2 - \frac{p_2}{p_1}b_1 \leq 0 \iff \frac{p_1}{p_2} \leq \frac{b_1}{b_2}.$$

This is the same condition we found in the preceding.

Other cases: Left to the reader as an exercise.

2 Quasilinear Utility

Consider the problem

$$\begin{aligned} V(p,w) = \max_{x_1,x_2} \; & e^{x_1}x_2 \\ & p_1x_1 + p_2x_2 \leq w \\ & x_1, x_2 \geq 0. \end{aligned}$$

Notice in this problem that the gradient of the utility $u(x_1, x_2) = e^{x_1}x_2$ is strictly positive at all x_1, x_2. So the objective is strictly increasing. Thus at the optimum it must be that $p_1x_1^* + p_2x_2^* = 0w$. Next, we can also see that $x_1^* > 0$ since $x_2 = 0$ will make the utility 0 while any positive x_2 will yield a positive utility. Finally, note that x^* is unchanged when we apply an increasing transformation to the objective. Applying log yields the simpler problem:

$$\begin{aligned} V(p,w) = \max_{x_1,x_2} \; & x_1 + \log(x_2) \\ & p_1x_1 + p_2x_2 \leq w \\ & x_1, x_2 \geq 0. \end{aligned}$$

The objective is now *quasilinear* in good 1. Taking into account what we have argue above about the constraints yields the problem

$$V(p,w) = \max_{x_1,x_2} x_1 + \log(x_2)$$
$$p_1x_1 + p_2x_2 = w$$
$$x_1 \geq 0.$$

The Lagrangian is

The KKT conditions are

$$\mathcal{L}(x_1, x_2, \lambda, \mu) = x_1 + \log(x_2) + \lambda(w - p_1x_1 - p_2x_2) + \mu x_1.$$

The KKT conditions are:

$$\begin{aligned}
1 - p_1\lambda + \mu &= 0 \\
\frac{1}{x_2} - p_2\lambda &= 0 \\
p_1x_1 + p_2x_2 &= w \\
x_1 &\geq 0 \\
\lambda(w - p_1x_1 - p_2x_2) &= 0 \\
\mu x_1 &= 0 \\
\lambda, \mu &\geq 0.
\end{aligned}$$

Combining the first two equations (using the fact that $p_2 > 0$) yields

$$1 = \frac{p_1}{p_2x_2} - \mu_1.$$

Case 1: $\mathbf{x_1^* = 0}$. In this case $x_2^* = w/p_2 > 0$ due to the third KKT condition. Substitution yields

$$1 + \mu_2 = \frac{p_1}{w}.$$

Using $\mu_2 \geq 0$ yields

$$\frac{p_1}{w} \geq 1.$$

Case 2: $\mathbf{x_1^* > 0}$. Now $\mu_1 = 0$. So we have

$$x_2^* = \frac{p_1}{p_2}.$$

The third KKT condition then yields

$$\begin{aligned} x_1^* &= \frac{w}{p_1} - \frac{p_2}{p_1} x_1^* \\ &= \frac{w}{p_1} - \left(\frac{p_2}{p_1}\right)\left(\frac{p_1}{p_2}\right) \\ &= \frac{w}{p_1} - 1. \end{aligned}$$

3 Geometry of FOCs

Consider the general problem

$$\begin{aligned} V(p, w) = \max_{x_1, x_2} \; & u(x_1, x_2) \\ & p_1 x_1 + p_2 x_2 \leq w \\ & x_1, x_2 \geq 0. \end{aligned}$$

Suppose that u is increasing in both arguments and that at the optimum $p_1 x_1^* + p_2 x_2^* = w$. Then the KKT conditions reduce the geometric condition that the indifference curve is tangent to the budget line at the optimum. In the preceding problem, the indifference curves are defined through

$$x_1 + \log(x_2) = k.$$

If we assume that x_2 is a function of x_1 then we can... (next time).

Week 7, Tuesday

Faheem Gilani

Spring 2025

Preferences and Utility

A rational preference relation $\succ$ on a set X is a *complete* and *transitive* binary relation on X. This means:

1. (Completeness of $\succ$): for all $x, y \in X$, either $x \succ y$ or $y \succ x$ (or both).
2. (Transitivity of $\succ$): If $x \succ y$ and $y \succ z$ then $x \succ z$.

Lemma 0.1. Let X be a finite set. Then $\succ$ on X is complete and transitive if and only if there exists a function $u : X \to \mathbb{R}$ such that $x \succ y$ if and only if $u(x) > u(y)$. This function is referred to as a *utility function* representing $\succ$.

The proof of this result requires "sorting" the set X.

The preceding lemma extends to more general X. Of special interest is the following:

Lemma 0.2. $X = \mathbb{R}^n_+$ and $\succ$ on X is complete, transitive, and *continuous* if and only if there exists a continuous function $u : X \to \mathbb{R}$ such that $x \succ y$ if and only if $u(x) > u(y)$.

We say that $\succ$ on $\mathbb{R}^n_+$ is open if for all $x \in \mathbb{R}^n_+$, the set $U_x = \{y \in \mathbb{R}^n_+ \mid y \succ x\}$ is an open subset of $\mathbb{R}^n_+$. Note: this is included here for completeness and you don't have to worry about it too much.

The preceding justifies utility maximization.

Indifference Surfaces

If $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ then indifference surfaces (curves) are of the form

$$x_1^\alpha x_2^{1-\alpha} = u^0.$$

In this case we can solve for x_2:

$$x_2(x_1) = (u^0)^{\frac{1}{1-\alpha}} x_1^{\frac{\alpha}{1-\alpha}}.$$

We can obtain the slope of the indifference curve via $\frac{dx_2(x_1}{dx_1}$ by differentiating the preceding expression. More generally though we can implicitly differentiate the expression

$$u(x_1, x_2(x_1)) = u^0$$

via the chain rule. Doing so would obtain:

$$\partial_{x_1} u(x_1, x_2(x_1)) + \partial_{x_2} u(x_1, x_2(x_1)) \frac{dx_2(x_1)}{dx_1} = 0$$

Solving for $\frac{dx_2(x_1)}{dx_1}$ we would obtain

$$\frac{dx_2(x_1)}{dx_1} = -\frac{\partial_{x_1} u(x_1, x_2(x_1))}{\partial_{x_2} u(x_1, x_2(x_1))}.$$

Note: if we wanted $\frac{dx_1(x_2)}{dx_2}$ then we could redo the preceding calculation with the assumption that x_1 is a function of x_2 instead.

Recall that the slope of the budget line is $-p_1/p_2$. If

$$\frac{\partial_{x_1} u(w/p_1, 0)}{\partial_{x_2} u(w/p_1, 0)} > \frac{p_1}{p_2}$$

then the indifference curve is steeper than the budget line at $(w/p_1, 0)$. So $(x_1^*, x_2^*) = (w/p_1, 0)$. We can redo the preceding calculation to obtain a similar condition for when $(x_1^*, x_2^*) = (0, w/p_w)$.

We can obtain obtain the same result through the KKT conditions. Recall that the Lagrangian for this problem is

$$\mathcal{L}(x_1, x_2, \lambda, \mu_1, \mu_2) = u(x_1, x_2) + \lambda(w - p_1 x_1 - p_2 x_2) + \mu_1 x_1 + \mu_2 x_2.$$

The KKT conditions are:

$$\begin{aligned}
\partial_{x_1} u(x_1, x_2) - p_1\lambda + \mu_1 &= 0 \\
\partial_{x_2} u(x_1, x_2) - p_2\lambda + \mu_2 &= 0 \\
p_1 x_1 + p_2 x_2 &\leq w \\
x_1, x_2 &\geq 0 \\
\lambda(w - p_1 x_1 - p_2 x_2) &= 0 \\
\mu_1 x_1 &= 0 \\
\mu_2 x_2 &= 0 \\
\lambda, \mu_1, \mu_2 &\geq 0.
\end{aligned}$$

If $x^* = (0, w/p)$ then (since the KKT conditions are necessary), it must be that

$$\begin{aligned}
\partial_{x_1} u(0, w/p_1) - p_1\lambda + \mu_1 &= 0 \\
\partial_{x_2} u(0, w/p_1) - p_2\lambda &= 0 \\
\mu_1 &\geq 0.
\end{aligned}$$

Combining the first equation with the last inequality in the preceding yields

$$\begin{aligned}
\partial_{x_1} u(0, w/p_1) - p_1\lambda &\leq 0 \\
\partial_{x_2} u(0, w/p_1) - p_2\lambda &= 0.
\end{aligned}$$

Combining these two yields:

$$\frac{\partial_{x_1} u(0, w/p_1)}{p_1} \leq \frac{\partial_{x_2} u(0, w/p_1)}{p_2},$$

which is equivalent to what we had before.

Geometry of $\mathbb{R}^n$

- The length or *norm* of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, denoted $\|x\|$ is

$$\|x\| := \sqrt{x_1^2 + x_2^2 + \ldots x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

  This is the "straight line" distance between two points in $\mathbb{R}^n$ and it is calculated via the Pythagorean theorem.

- The dot product of two vectors $x, y \in \mathbb{R}^n$, denoted $x \cdot y$, is defined via

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

- We can use the dot product to rewrite norm in a quicker way:

$$\|x\| = \sqrt{x \cdot x}$$

- Norm and dot product are connected through:

$$x \cdot y = \|x\|\|y\| \cos(\theta).$$

  One proof of this involves applying the law of cosines. See notes. The proof in class used the Pythagorean theorem to show that

$$\|y\| \cos(\theta) = \frac{x \cdot y}{\|x\|}.$$

  In this proof, consider a right triangle with hypotonuse of length $\|y\|$ and an adjacent side parallel to x. The adjacent side has length $\|y\| \cos(\theta)$ and it is parallel to the vector x. So we can write $\|y\| \cos(\theta) = t\|x\|$ for some t. The Pythagorean theorem then tells us that

$$\begin{aligned} \|y\|^2 &= \|tx\|^2 + \|y - tx\|^2, \\ y \cdot y &= t^2 x \cdot x + y \cdot y - 2tx \cdot y + t^2 x \cdot x, \\ 0 &= 2t^2 x \cdot x - 2tx \cdot y, \end{aligned}$$

  from which we find that (if $y \neq \mathbf{0}$) then

$$t = \frac{x \cdot y}{x \cdot x} = \frac{x \cdot y}{\|x\|^2}.$$

  Thus

$$\begin{aligned} \|y\| \cos(\theta) &= t\|x\| \\ &= \frac{x \cdot y}{\|x\|^2}\|x\| \\ &= \frac{x \cdot y}{\|x\|} \end{aligned}$$

  , as desired.

- Let $x \in \mathbb{R}^n$. We say that $y \in \mathbb{R}^n$ is *orthogonal* to x, denoted $y \perp x$ if $x \cdot y = 0$. If $S \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ then its *orthogonal complement* is the set $S^\perp := \{y \mid y \cdot x = 0 \text{ for all } x \in S\}$. The possible subspaces of $\mathbb{R}^n$ are: $\{\mathbf{0}\}$, any line through the origin, any $2-d$ plane through the origin, any $3-d$ plane through the origin (if $n \geq 3$),..., etc.

  Recall that the budget set is $\{x \in \mathbb{R}^n_+ \mid p \cdot x = w\}$. This is a translation of the orthogonal complement to p.

Week 8, Thursday

Faheem Gilani

Spring 2025

Direction of Fastest Increase

In $\mathbb{R}$. In $\mathbb{R}$, the gradient of a function u at x^0 is $u'(x^0)$. Gradients are vectors, which have directions. We show that the gradient in $\mathbb{R}$ points in the direction of fastest increase. Note that there are only two directions in $\mathbb{R}$: positive and negative. The equation of the tangent line at $(x^0, u(x^0))$ to a function $u : \mathbb{R} \to \mathbb{R}$, is $T(x) = u(x^0) + u'(x^0)(x - x^0)$. Since u is well approximated at x^0 by its tangent line, if $u'(x^0) > 0$ then u is strictly increasing at x^0 and If $u'(x^0) < 0$ then u is strictly decreasing. In either case, the gradient points in the direction of greatest increase. If $u'(x^0) = 0$ then we are at a maximum. The 0 vector has an undefined direction (or perhaps in all directions).

In $\mathbb{R}^\times$. The tangent plane to a function $u(x)$ at the point x^0 is

$$T(x) = u(x^0) + Du(x^0) \cdot (x - x^0).$$

Note that $T(x^0 + v) - T(x^0) = D(x^0) \cdot v$ due to $T(x^0) = u(x^0)$. Suppose u is a unit vector so that $\|v\| = 1$. Amongst all v, the direction of fastest increase along the tangent plane solves

$$\max_{\|v\|=1} Du(x^0) \cdot v.$$

To solve this, recall that

$$\begin{aligned} Du(x^0) \cdot v &= \|Du(x^0)\|\|v\| \cos(\theta) \\ &= \|Du(x^0)\| \cos(\theta)\|, \end{aligned}$$

where θ is the angle between $D(x^0)$ and v. The expression $\|D(x^0)\| \cos(\theta)\|$ is maximized when $\cos(\theta) = 1$, which occurs when $\theta = 0$. That is, when v is parallel (in the same direction) as the gradient. The preceding also shows that if $\theta = \pi/2, 3\pi/2$, then $D(x^0) \cdot v = 0$ so any direction orthogonal to $Du(x^0)$ leads to no change along the tangent plane.

Since differentiable u are well approximated at a point by their tangent plane at that point, we find that the direction of greatest increase at x^0 along u is $Du(x^0)$ and the direction of no increase is any direction orthogonal to $Du(x^0)$.

For example, let $u(x) = x_1^{1/2} + x_2^{1/2}$. Then $Du(x) = \left(\frac{1}{2}x + 1^{-1/2}, \frac{1}{3}x_2^{-2/3}\right)$. The equation of the tangent plane at $x^0 = (25, 27)$ is therefore

$$\begin{aligned} T(x) &= u(x^0) + Du(x^0)(x - x^0) \\ &= 8 + (\frac{1}{10}, \frac{1}{27}) \cdot (x_1 - 25, x_2 - 27) \\ &= 8 + \frac{1}{10}(x_1 - 25) + \frac{1}{27}(x_2 - 27). \end{aligned}$$

Suppose we move to the point $(30, 31)$. The vector corresponding to this displacement is $(5, 4)$. Then $T(30, 31) - T(25, 27) = \frac{1}{2} + \frac{4}{27}$ (recall that $T(x) - T(x^0) = Du(x^0)(x - x^0)$) .

In this example, the direction of greatest increase at $x^0 = (25, 27)$ is $(1/10, 1/27)$.

KKT Conditions:

Consider the problem

$$V(b) = \begin{cases} \max_{x \in \mathbb{R}^n} f(x) \\ g(x) \leq b, \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^k$, $b \in \mathbb{R}^k$.

03.27.2025, Thursday

Faheem Gilani, Hilal Taşkan
Spring 2025

Topics

1. Quiz Solutions

1 Quiz Solutions

Consider the problem

$$e(p_1, p_2, 1) = \min_{x_1 x_2 \geq 1} p_1 x_1 + p_2 x_2.$$

Part 1: Then $x_2 = 1/x_1$ so the MRS is $x_2'(x_1) = -x_1^{-2}$. Setting this equal to $-p_1/p_2$ yields

$$x_1^h = \sqrt{\frac{p_2}{p_1}}.$$

Substituting this into $x_1 x_2 = 1$ yields

$$x_2^h = \sqrt{\frac{p_1}{p_2}}.$$

Part 2: Plugging these back into $p_1 x_1 + p_2 x_2$ yields

$$e(p_1, p_2, 1) = 2\sqrt{p_1 p_2}.$$

Part 3: The final part of the quiz asked the following. If at $t = 0$ prices are p^0 and $t = 1$ prices are p^1 then the amount of wealth that leaves the consumer at the same level of happiness in period 2 as in period 1 is given by

$$w = e(p_1^1, p_2^1).$$

The wealth multiplication factor is

$$\frac{e(p_1^1, p_2^1)}{e(p_1^0, p_2^0}$$

2 Using Duality for Calculations

utility Maximization: Consider the problem

$$\max_{p_1x_1+p_2x_2\leq w, x_1,x_2\geq 0.} 10x_1^{9/10} + 100x_2^{9/10}.$$

The objective (utility) function is increasing so the budget constraint binds. Furthermore, the nonnegativity constraints are slack: as we approach 0 consumption of a good, the bang for the buck of the other good becomes infinite . So the Lagrangian is

$$\mathcal{L}(x_1, x_2, \lambda) = 10x_1^{9/10} + 100x_2^{9/10} + \lambda(w - p_1x_1 - p_2x).$$

The first order conditions simplify to (as is usual):

$$\begin{aligned} \frac{9x_1^{-1/10}}{90x_2^{-1/10}} &= \frac{p_1}{p_2} \\ p_1x_1 + p_2x_2 &= w. \end{aligned}$$

Solving for x_1 in the first equation yields $x_2 = \kappa x_1$ where

$$\kappa = \left(10\frac{p_1}{p_2}\right)^{10}.$$

Substituting this into the second constraint yields

$$x_1^* = \frac{w}{p_1 + p_2\kappa}.$$

Substituting this into the expression for x_2 yields

$$x_2^* = \frac{\kappa w}{p_1 + p_2\kappa}.$$

The value function is

$$\begin{aligned} V(p, w) &= 10\left(\frac{w}{p_1 + p_2\kappa}\right)^{9/10} + 100\left(\frac{\kappa w}{p_1 + p_2\kappa}\right)^{9/10} \\ &= w^{9/10}M, \end{aligned}$$

where

$$M = 10\left(\frac{1}{p_1 + p_2\kappa}\right)^{9/10} + 100\left(\frac{w}{p_1 + p_2\kappa}\right)^{9/10}.$$

Expenditure Function from Indirect Uility through Duality: We want to use Duality and the expression for the indirect utility function that we calculated in the preceding in order to calculate the expenditure function. Recall the duality identity:

$$V(p_1, p_2, e(p_1, p_2, u^0)) = u^0.$$

Substituting the expression for the indirect utility function that we calculated above yields

$$e(p_1, p_2, u^0)^{9/10} M = u^0,$$

from which

$$e(p_1, p_2, u^0) = \left(\frac{u^0}{M}\right)^{10/9}$$

It can be verified that this is the expenditure function by calculating the Hicksian demand directly and then getting the expenditure function.

Indirect Utility From Expenditure through Duality: Recall that

$$e(p_1, p_2, V(p_1, p_2, w)) = w.$$

Substituting the expression for theexpenditure function into this yields

$$\left(\frac{V(p_1, p_2, w)}{M}\right)^{10/9} = w.$$

Solving for $V(p_1, p_2, w)$ yields the same expression as before.

03.27.2025, Thursday

Faheem Gilani, Hilal Taşkan
Spring 2025

Topics

1. Quiz Solutions
2. Slutsky Equations
3. Saddle Points
4. Envelope Theorem
5. Slutsky Equations (Derivation)

1 Quiz Solutions

Let $u(x_1, x_2) = x_1 + x_2$.

Question 1: The MRS is $\frac{\partial_{x_1} u(x_1,x_2)}{\partial_{x_2} u(x_1,x_2)} = -1$. Indifference Curves are of the form $k = x_1 + x_2$. This can be written as $x_2 = k - x_1$ (these are lines).

Question 2: We are asked to show that $e(p_1, p_2, u^\circ) = u^\circ \min\{p_1, p_2\}$ by using the fact that for $p_1 \neq p_2$,

$$h(p, u^\circ) = \begin{cases} (u^\circ, 0) & \text{if } p_1 = \min\{p_1, p_2\} \\ (0, u^\circ) & \text{if } p_2 = \min\{p_1, p_2\} \end{cases}.$$

Notice that when $p_1 \neq p_2$ we have

$$\begin{aligned} e(p_1, p_2, u^\circ) = p \cdot h &= \begin{cases} p \cdot (u^\circ, 0) & \text{if } p_1 = \min\{p_1, p_2\} \\ p \cdot (0, u^\circ) & \text{if } p_2 = \min\{p_1, p_2\} \end{cases} \\ &= \begin{cases} p_1 u^\circ & \text{if } p_1 = \min\{p_1, p_2\} \\ p_2 u^\circ & \text{if } p_2 = \min\{p_1, p_2\} \end{cases} \\ &= u^\circ \min\{p_1, p_2\}. \end{aligned}$$

When $p_1 = p_2$, the consumer is indifferent between any combination of bundles that achieves utility u°. The result follows.

Question 3: Recall that $e(p, V(p, w)) = w$. Using the expression for the expenditure function above we have $V(p, w) \min\{p_1, p_2\} = w$, from which we find that $V(p, w) = \frac{w}{\min\{p_1, p_2\}}$.

Question 4: At $t = 0$ the indirect utility $V((2, 3), w^0) = 25$. Using the expression for indirect utiltiy we find that $\frac{w^0}{\min\{2,3\}} = \frac{w^0}{2}$. So $w^0 = 25 \times 2 = 50$. To be as well off in period 1, the consumer must have $e(p^1, 25) = 25 \min\{3, 5\} = 75$. So w^0 must be multiplied by 3/2.

2 Slutsky Equations

We will show that

$$\partial_{p_i} x_i^*(p, w) = \partial_{p_i} h_i(p, u^\circ) - x_i^*(p, w) \partial_w x_i(p, w),$$

where u° is the indirect utility $V(p, w)$. This is the *Slutsky Equation* for good i. If

- $\partial_{p_i} x_i(p, w) < 0$ then good i is a normal good.
- $\partial_w x_i(p, w) < 0$ then good i is an inferior good.
- If $\partial_{p_i} x_i(p, w) > 0$ then good i is a Giffen good.

3 Saddle Points

Suppose (x^*, λ^*) is a saddle point of the Lagrangian. That is, x^* solves $\max_x \mathcal{L}(x, \lambda^*)$ and λ^* solves $\min_{\lambda \geq 0} \mathcal{L}(x^*, \lambda)$. Here, we show that x^* solves $\max f(x)$ subjected to the constraints $g(x) \leq b$

First, we show that $b - g(x^*) \geq 0$. Indeed, suppose not. Then there exists j such that $b_j - g_j(x^*) < 0$. Consider $\lambda^*(b - g(x^*) = \sum_{i \neq j} \lambda_i^*(b_i - g_i(x^*)) + \lambda_j^*(b_j - g_j(x^*))$. Since $(b_j - g_j(x^*))$ we can $\mathcal{L}(x^*, \lambda^*)$ smaller by making λ_j^* larger. So λ^* does not solve $\min_{\lambda \geq 0} \mathcal{L}(x^*, \lambda)$, which means that (x^*, λ^*) is not a Saddle Points.

Next, we show that $\lambda^*(b - g * x^*) = 0$. If not, then due to the preceding, there is an index j such that $b_j - g_j(x^*) > 0$. So if $\lambda_j^* > 0$, we could make $\mathcal{L}(x^*, \lambda^*)$ smaller by setting $\lambda_j^* = 0$.

Finally, we show that x^* solves $\max f(x)$ subjected to the constraints $g(x) \leq b$. Indeed, let λ^* be such that (x^*, λ^*) is a saddle point of $\mathcal{L}(x, \lambda)$. Then

$f(x^*) + \lambda^*(b - g(x^*)) > f(x) + \lambda^*(b - g(x^*))$. From the preceding, we know that $\lambda^*(b - g(x^*)) = 0$. The conclusion follows.

Concave Objective with Convex Constraints: Now suppose f is concave and each g_j is convex. If $D_x\mathcal{L}(x^*, \lambda^*) = 0$ then x^* solves $\max_x \mathcal{L}(x, \lambda^*)$ since the latter is a an unconstrained optimization problem with a concave objective.

If $\lambda^* \geq 0$, $b - g(x^*) \geq 0$ and $\lambda^*(b - g(x^*)) = 0$ then λ^* solves $\min_{\lambda \geq} \mathcal{L}(x^*, \lambda)$. Thus if (x^*, λ^*) solves the KKT conditions, we find that (x^*, λ^*) is a saddle point of $\mathcal{L}$. So the KKT conditions in this case are both necessary and sufficient.

4 Envelope Theorem

Consider

$$V(b) = \max f(x) \;\; \text{st.} g(x) \leq b.$$

Then $\mathcal{L}(x, \lambda) = f(x) + \lambda(b - g(x))$. Recall that the Envelope theorem in this context says that $\partial_{b_i} V(b) = \partial_{b_i} \mathcal{L}(x, \lambda)_{(x,\lambda)=(x^*,\lambda^*)}$ where (x^*, λ^*) is a saddle point of $\mathcal{L}$. In this case, $V(b) = \max_x (f(x) + \lambda^*(b - g(x))$ so $DV(b) = \lambda^*$.

5 Slutsky Equations (Derivation)

To derive the Slutsky Equations, recall that $h(p, u^\circ)$ is the consumption bundle that solves

$$\min p \cdot x \;\; \text{st. } u(x) \geq u^\circ.$$

Fix u° and let p' and p'' be two price levels and let $h' = h(p', u^\circ), h'' = h(p'', u^\circ)$ be the respective Hicksian demands. From the homework we know that $(p' - p'') \cdot (h' - h'') \leq 0$. In particular, let $p' = p$ and $p'' = p + \epsilon e_i$ where $\epsilon > 0$. Then $(p' - p'') \cdot (h' - h'') = \epsilon(h_i(p + \epsilon, u^\circ) - h_i(p, u^\circ))$. So $\epsilon(h_i(p + \epsilon, u^\circ) - h_i(p, u^\circ)) \leq 0$. Dividing by ϵ^2 yields

$$\frac{h_i(p + \epsilon, u^\circ) - h_i(p, u^\circ)}{\epsilon} \leq 0.$$

If h_i is differentiable then as $\epsilon \downarrow 0$ we

04.03.2025, Thursday

Faheem Gilani
Spring 2025

Topics

1. Slutsky Equations using Envelope Theorem
2. Production

1 Slutsky Equations using Envelope Theorem

Recall the Envelope theorem: If x° is optimal and $\theta = \theta^\circ$ and the value function $V(\theta)$ is differentiable at θ° then $\partial_{\theta_i} = \partial_{\theta_i}\mathcal{L}(x^*, \theta^*, \lambda^*)$. When there are no constraints (or when none of the constraints bind), we have $\partial_{\theta_i}\mathcal{L}(x^*, \theta^*, \lambda^*) = \partial_{\theta_i} f(x^*, \theta^*)$, where f is the objective functions.

2 Production

Consider the problem

$$C(w, y) = \min w \cdot x \quad \text{st.} f(x) \geq y,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a production function, w is the cost of each factor x_i and y is the amount of the good being produced. $x^*(w, y)$ is known as the conditional factor demand(s). By the envelope theorem, $\partial_{w_i} C(w, y) = x_i^*$

Suppose there are n inputs and m outputs. The set of feasible input-output (netput) vectors in $\mathbb{R}^{n+m}$ is $Y \subset \mathbb{R}^{n+m}$. If $x \in Y$ then $x = (x^{in}, x^{out})$ with $x^i n \leq 0$ and $x^{out} \geq 0$. To respect physical reality, we assume that $Y \cap \mathbb{R}_+^{n+m} = \{0\}$

The profit function is

$$\Pi(w, p) = \max_{y \geq 0} p \cdot y - c(w, y).$$

If there is one good being produced via a production function then the preceding is

$$\Pi(w, p) = \max_{x \geq 0} pf(x) - w \cdot x.$$

We can write the entire profit maximization problem in a more abstract form:

$$\Pi_Y(p) = \max_{y \in Y} p \cdot y.$$

where $y = (y^{in}, y^{out})$ is the netput vector (the first n vectors are inputs and the next m are outputs) and $p = (p^{in}, p^{out})$ is the vector of prices (thet first n are the prices corresponding to the inputs and the next m are the prices corresponding to outputs).

Law of Supply and Demand: If the price of an output good increases then its production weakly increases. If the price of an input good increases then its production weakly decreases. The argument is the same as for the law of compensated demand (see previous homework). Also recall the same discussion when we were discussing super/sub-modularity

04.08.2025, Tuesday

Faheem Gilani
Spring 2025

Topics

1. Decentralization with Calculus
2. Decentralization

1 Decentralization with Calculus

To solve production problems we can do one of the following:

1. Find $c(y, w) = \min w \cdot x$ such that $f(x) \geq y$
2. Solve $\Pi(p, w) = \max_{x \in \mathbb{R}^n} pf(x) - w \cdot x$.

The solution to one of these can get us the solution to the other.

Example: Suppose the production function is $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$. The setting is: there are two firms producing the same good y using the same material. They differ only in their productivity. Suppose that the cost of each input good is the same for each of the firms. location. Then the cost minimization problem is

$$c(y, w) = \min_{x_1, x_2} wx_1 + wx_2 \;\; \text{st.} f_1(x_1) + f_2(x_2) \geq y.$$

The Lagrangian is $\mathcal{L}(x_1, x_2, \lambda) = wx_1 + wx_2 + \lambda(y - f_1(x_1) + f_2(x_2))$. The FOCs are

$$w = \lambda f_j'(x_j) \;\; \text{for } j = 1, 2$$
$$y \leq f_1(x_1) + f_2(x_2)$$

By the envelope theorem, $\lambda^* = \partial_y \mathcal{L} = \partial_y c(y, w)$. So

$$f_j'(x_j) = \frac{w}{\partial_y c(y, w)},$$

for $j = 1, 2$.

To calculate the supply y after solving for the cost function, we can solve

$$\max_{y \geq 0} py - c(w, y)$$

The Takeaway:

1. In an ideal world, an omniscient central planner can coordinate the production of all resources in the world in order to achieve the "best possible" outcome. In the preceding it was the maximization of total profit but the objective can be more complex. The setting can also be more complex.

2. In the real world nobody really has enough information to be this omniscient. So relying on such a central planner is usually a nonstarter. Even if such an entity existed, there is no guarantee that they would solve the complicated maximization problem that arises. Going back to our example: even the coordination of all the resources in a small setting like UT-Austin in order to achieve some social optimum is a complicated integer programming problem that would probably take years for a super computer to solve (even if it could be written down).

3. The result here is that by setting prices appropriately, we can achieve the same outcome as in the first case above but this time by allowing each firm to solve their own optimization problem without caring about what the other firms are doing and magically we achieve the same (caveat) solution as what the omniscient planner would choose but without requiring that the omniscient planner exist. The difference is that this is a considerably easier problem to solve. This is a special case of a more general family of such results.

2 Decentralization

Homework Problem. Refer to your notes from the review session. Additional info: the right price equalizes the maginal benefit of the last unit used.

04.10.2025, Thursday

Faheem Gilani, Hilal Taşkan

Spring 2025

Topics

1. Coordinating Production
2. Edgeworth Box: perfect substitutes.

1 Coordinating Production

Centralized Approach Suppose there are M firms, each producting the same good. Index each firm's production function by f^i, $i \in \{1, \ldots, M\}$. There are n total input goods in the economy. The centralized problem of allocating a total resource amounts $X \in \mathbb{R}^n$. The maximization problem is:

$$\max_{(x^i)_{i=1}^n, x^i \in \mathbb{R}^n} \sum_{i=1}^{M} f^i(x^i)$$
$$\sum_{i=1}^{M} x_i \leq X.$$

If each f^i is increasing then the resource constraint binds. In that case we can study the Lagrangian: $\mathcal{L}(x, \lambda) = \sum_{m=1}^{M} f^i(x^i) + \lambda \cdot (X - \sum_{m=1}^{M} x^m)$. The FOCs are:

$$\frac{\partial f^m(x^m)}{\partial x_i} = \lambda_i$$
$$\sum_{m=1}^{M} x_i^m = X_i.$$

Decentralized Approach Suppose instead that each firm can buy inputs at price $w \in \mathbb{R}^n$ and the market price of the goods produced is $p \in \mathbb{R}$. Then each firm solves

$$\max_{x^m} p f^m(x^m) - w x^m.$$

The FOCs are

$$p\frac{\partial f^m}{\partial x_i} - w_i = 0$$
$$\frac{\partial f^m}{\partial x_i} = \frac{w_i}{p}.$$

We obtain the same FOCs as in the central problem when $\frac{w_i}{p} = \lambda_i$. At these relative prices, $\sum_m x^m = X$ (the demand is equal to the supply). In a competitive price taking market setting, we expect these relative prices to be obtained due to market forces: inputs that are over demanded will increase in prices and goods that are underdemanded will decrease in prices. There are two upshots to this approach:

1. Solving the centralized problem requires firm specific knowledge (their production function and costs) that may be hard to elicit by a central planner. The decentralized approach allows each firm to take prices as given and leverage their personalized knowledge directly.
2. The decentralized approach is more responsive to changes in underlying conditions (like supply shocks) are summarized through prices. The central planner instead would need to recollect all the data again following a change in supply for a particular good, for example.

2 Edgeworth Box

Suppose there are two people: person A and person B. The endowments are:

1. $e^A = (17, 3)$
2. $e^B = (20, 31)$.

The total endowment is $e = (37, 34)$. Let the x axis be good 1 and the y axis be good 2. The Edgeworth box has x axis running from 0 to 37 and the y axis running from 0 to 34.

2.0.1 Linear Utility

Suppose that the utilities are

$$u^A(x_1, x_2) = \frac{7}{3}x_1^A + x_2^A$$
$$u^B(x_1, x_2) = \frac{1}{4}x_1^B + x_2^B$$

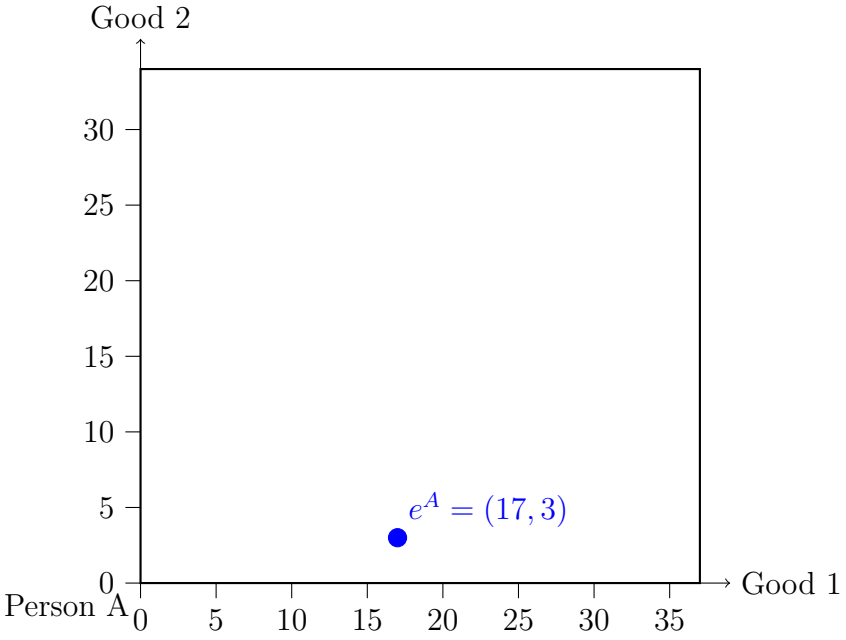


Figure 1: Edgeworth box for this economy

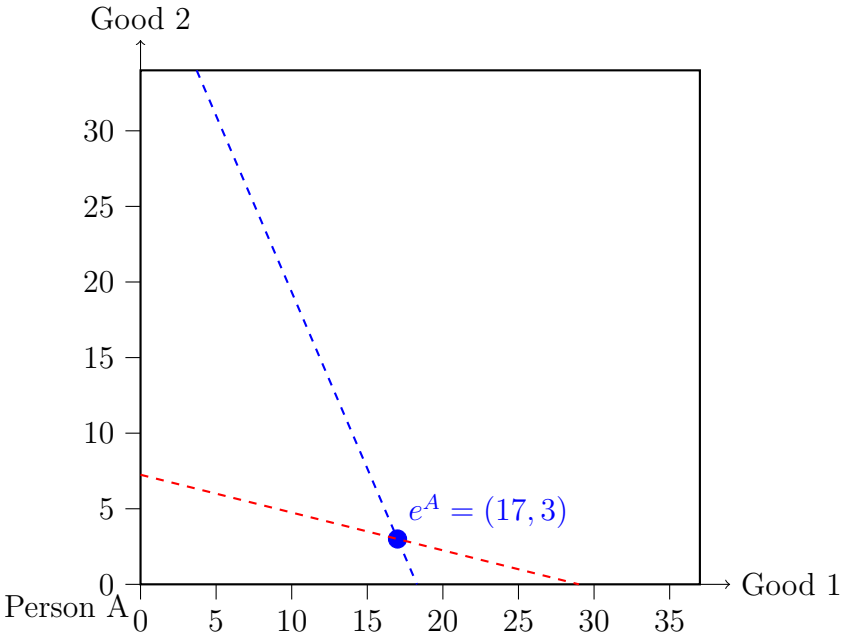


Figure 2: Indifference curves through the endowment. Blue is A's and red is B's

The homework contains further problems related to this.

04.17.2025, Thursday

Faheem Gilani

Spring 2025

Topics

1. The Problems with Externalities
2. Pigon's Solution
3. The create-a-market solution

1 Fishing

N firms, each picks $q_n \geq 0$ boats to send to the sea to fish. The cost per quantity of boat is c. Let $q = \sum_{n=1}^{N} q_n$ be the total fleet size. Let $v(q)$ be the "per boat catch when the total fleet size is q and assume that $v(0) > c$, $v'(q) < 0$ and $v''(q) \leq 0$. Let $q_{-i} = \sum_{j \neq i} q_j$. Then firm i's best response to q_{-i} satisfies

$$\begin{aligned} \pi_i(q_i, q_{-i}) &= q_i v(q) - c q_i \\ &= q_i v(q_i + q_{-i}) - c q_i. \end{aligned}$$

To characterize the optimum $q_i^*(q_{-i})$, notice that the objective is concave since

$$\partial_{q_i}^2 \pi(q_i, q_{-i}) = 2v'(q_i + q_{-i}) + q_i v''(q_i + q_{-i}) < 0$$

due to $q_i \geq 0$, $v' > 0$ and $v'' \leq 0$. So the KKT conditions are sufficient.

We show that if $q_i^* > 0$ for all i then $q_i^* = q_j^*$ for all firms i, j. To establish this, notice that if $q_i^* > 0$ for all i then the KKT conditions become

$$v(q^*) + q_i^* v'(q^*) = c$$

for all i. Without loss of generality, suppose $q_i^* > q_j^*$. Then $v(q^*) + q_i^* v'(q^*) > v(q^*) + q_j^* v'(q^*)$. This violates the fact that they must both be equal to c. So any Nash equilibrium with $q_i^* > 0$ must involve $q_i^* = q_j^*$ for all firms i and j.

Next we show that it cannot be the case that $q_i^* = 0$ for all i. If this occured then it would be necessary that $\partial_{q_i} \pi(0, 0) \leq 0$. However,

$$\partial_{q_i} \pi(0, 0) = v(0) - c > 0$$

by assumption.

Finally, we show that it cannot be the case that in equilibrium $q_i^* = 0$ while $q_j^* > 0$. If that were the case then it would be necessary that

$$v(q^*) \leq c$$

while

$$v(q^*) + q_j^* v'(q^*) = c.$$

Note that the first equation is firm i's FOC at 0 and the second is firm $j's$ at $q_j^* > 0$. Combining these two we would obtain

$$v(q^*) \leq v(q^*) + q_j^* v'(q^*).$$

Equivalently,

$$0 \leq q_j^* v'(q^*).$$

Since $q_j^* > 0$, $q^* > 0$ and since $v' < 0$, it must be that $q_j^* v'(q^*) < 0$ so the preceding cannot hold. Thus it cannot be the case that in equilibrium we have $q_i^* = 0$ for some i and $q_j^* > 0$ for some j.

Thus we have shown that the Nash equilibrium is $q_i^* > 0$ for all i and $q_i^* = q_j^*$ for all i, j and such quantities are determined through the FOCs $v(q^*) + q_i^* v'(q^*) = c$. We can simplify this a bit by adding all FOCs and obtaining $Nv(q^*) + q^* v'(q^*) = Nc$.

Finally, a maximizer exists. To see this, note that $v(q^*)q_i^*$ is bounded that there exists some q_c such that $v(q^*)q_c = 0$.

04.22.2025, Tuesday

Faheem Gilani
Spring 2025

Topics

1.

1 Fishing and Externalities (continued)

There are N firms. Firm i chooses an amount q_i of boats to send out to sea to fish. If each firm i sends out q_i then firm i's profit is

$$\Pi_i(q_i, q_{-i}) = q_i v(q_i) - cq_i,$$

where $q_{-i} = (q_j)_{j=1, j\neq i}^n$ and $q = \sum_{i=1}^n q_i$. Here $v(0) > c > 0$ and $v' > 0$ and $v'' \leq 0$. Looking at the best responses, the FOCs for interior Nash are

$$v(q) + q_i v'(q) = c.$$

Claim: If $(q_1^*, q_2^*, \ldots, q_n^*)$ is a Nash equilibrium of the preceding game then $q_j^* > 0$ for all j and $q_i^* = q_j^*$ for all i, j. The steps to establishing these claims is to show:

1. there exists j such that $q_j^* > 0$.
2. $q_i^* = q_j^*$ for all i, j.
3. For all j, $q_j > 0$.

To establish the first claim: Pick player j and suppose that $q_i = 0$ for all $i \neq j$. Then Player i's profit from choosing q_j is

$$\pi_j(q_j, 0) = q_j v(q_j) - cq_j.$$

Notice that $\partial_{q_j}\pi_j(0,0) = v(0) - c > 0$. But the necessary condition for $q_j = 0$ to be an optimum is that $\partial_{q_j}\pi_j(0,0) \leq 0$. So $q_j = 0$ is not a best response to all other players playing 0. Consequently, there is some j such that $q_j^* > 0$.

To establish the second claim, take two firms $i \neq j$. Suppose $q_i^*, q_j^* > 0$. Then through their FOCs we see that

$$v(q^*) + q_i^* v'(q^*) = c = v(q^*) + q_j^* v'(q^*).$$

Due to $q_i^*, q_j^* > 0$, the preceding implies that $q_i^* = q_j^*$.

Finally, we show that $q_i^* = q_j^*$ for all i, j. To do this we show that it is not possible for some firms $i \neq j$ to have $q_i^* > 0$ while $q_j^* = 0$. Then

$$\partial_{q_j} \pi_j(0, q_{-j}) = v(q^*) - c > 0. \tag{1}$$

From firm i's FOC (recall that $q_i^* > 0$) we know that

$$v(q^*) + q_i^* v'(q^*) = c.$$

Since $q_i^* > 0$ and $v' < 0$ we find that $q_i^* v'(q^*) < 0$. Since $c > 0$ we find that $v(q^*) > c$. Using this in Equation 1 we find that $\partial_{q_j} \pi_j(0, q_{-j}) > 0$, which contradicts the necessary conditions for optimality for for $q_j^* = 0$.

Thus we know that all $q_i^* > 0$. Adding all of the FOCs yields

$$Nv(q^*) + q^* v'(q^*) = Nc. \tag{2}$$

Here we have $q^* = \sum_{i=1}^n q_i^*$. Dividing Equation 2 by N yields

$$v(q^*) + \frac{q^*}{N} v'(q^*) = c. \tag{3}$$

We claim that as $N \uparrow \infty$, the surplus goes to 0 and that q^* increases in N.

To establish the first claim, note that as $N \uparrow \infty$, $q^*/N \downarrow 0$ while $v'(q)$ remains bounded. So the FOCs become $v(q^*) = c$. That is, the surplus goes to 0.

To establish the second claim, implicitly differentiating Equation 2 (treating q^* as a function of N) we find:

$$v(q^*(N)) + Nv'(q^*(N)) \frac{dq^*(N)}{dN} + \frac{dq^*(N)}{dN} v'(q^*(N)) + q^*(N) v''(q^*(N)) \frac{dq^*(N)}{dN} = 0.$$

Solving for $\frac{dq^*(N)}{dN}$ yields

$$\frac{dq^*(N)}{dN} = -\frac{v(q^*(N))}{Nv'(q^*(N)) + v'(q^*(N)) + q^*(N) v''(q^*(N))} > 0$$

due to the denominator being negative and the numerator being positive. Note that in the preceding we assumed N was continuous for differentiation and then established that $q^*(N)$ is increasing for all positive real N, not just natural N.

Another way to establish the preceding is the following. Define

$$f(q, N) = \int_0^q [v(x) + Nxv'(x)]\, dx - cq.$$

By the fundamental theorem of calculus, $\partial_q f = v(q) + Nv(q) - c$, $\partial^2_{qq} f \leq 0$ (as established before) and $\partial^2_{qN} = Nv(q) > 0$. So f is concave in q and strictly supermodular in q and N. Let $q^*(N)$ solve

$$\max_{q>0} f(q, N)$$

By Topkis', $q^*(N)$ is strictly increasing in N. Since the $q^*(N)$ for this problem satisfies the same FOCs as the $q^*(N)$ in the preceding, we find that $q^*(N)$ is increasing in that case as well.

2 Fishing Problem with Tax

Now suppose there is a tax $\tau \geq 0$ per unit of boat sent out. In particular, firm i's profit function is

$$\pi_i(q_i, q_{-i}) = q_i v(q) - (c + \tau) q_i.$$

The same analysis goes through but by treating c in the preceding as $c + \tau$. The FOCs aggregate to:

$$v(q^*) + \frac{1}{N} q^* v'(q^*) = c + \tau.$$

As N goes to ∞, q^* solves $v(q^*) = c + \tau$ so there is again no surplus and the industry is ruined.

Instead, suppose we give each firm permits to run q_s/N boats where q_s is the socially optimal amount. That is, q_s solves

$$\max_q (qv(q) - cq)$$